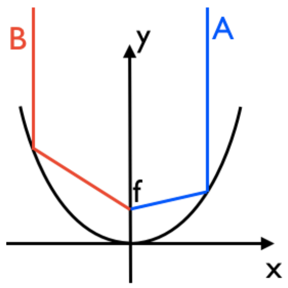
Werkcollege, Observational Astronomy 2016/2017
Week 2 (23 Nov 2016)

These are the assignments for the second week of the course *Observational Astronomy*. All students are required to hand in the answers to **Problem 2.2** below at a time agreed upon with the teaching assistant.
Once all hand-in assignments have been approved, the student receives 20 of the 100 points on which the final grade will be based. The remaining part of the grade will be based on the final project report.

2.1 Spherical and paraboloidal surfaces

a. The drawing below shows a paraboloidal surface, $y = ax^2$. Demonstrate that all parallel light rays from a distant source travel the same distance to the point $(0, f) = (0, 1/4a)$.

Hint: Consider how far a light ray has to travel from some distant point (x, y_0) to a point $(0, f)$. Then show that this distance is independent of x for $f = 1/4a$.

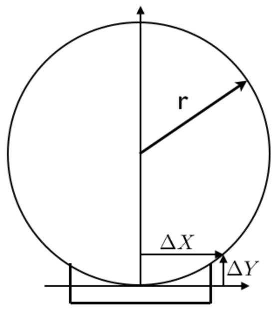


In the limit of long f-ratios, the shape of a paraboloid can be approximated by a spherical surface. This has the advantage that spherical surfaces are easier (and therefore cheaper) to produce, and do not suffer from coma.

b. In the figure below, consider the vertical position of the mirror surface ΔY as a function of distance ΔX from the optical axis. Assume that the mirror surface is part of a spherical surface with radius r. Show that the mirror surface can be approximated by the relation

$$\Delta Y \approx \frac{1}{2r}(\Delta X)^2$$

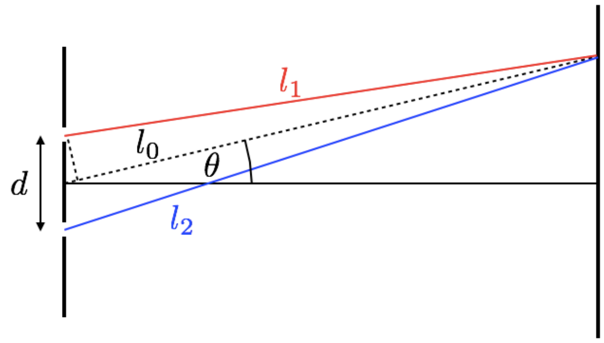
Hint: you may need to make use of the approximation $\sqrt{1+\delta} \approx 1 + \frac{1}{2}\delta$ for small δ.



2.2 Interference pattern in single slit (*hand-in assignment*)

Recap: the double-slit experiment

In the lecture we calculated the interference pattern in the classical double-slit experiment. We repeat the calculation here. As illustrated in the figure below, the basic geometry is that a plane wave (as would be produced by a point source at infinite distance from the slits) enters from the left, passes through two narrow slits separated by a distance d, and then produces an interference pattern on the screen at the right.



We make use of the *Fraunhofer approximation*, which simply means that we assume that the distance between the slits and the screen is very large compared to the separation between the slits ($l_0 \gg d$), so that the light "rays" along l_1 and l_2 can be considered parallel. (In a real telescope, this is still a valid approximation due to the focussing of the light by a lens or mirror.)

We shall assume that the angle θ is small, so that l_0 is independent of θ, and $\sin\theta \simeq \theta$. For light rays passing through the two slits, we then have

$$l_1 = l_0 - \frac{d}{2}\sin\theta \simeq l_0 - d\theta/2 \tag{2.2.1}$$

and

$$l_2 = l_0 + \frac{d}{2}\sin\theta \simeq l_0 + d\theta/2 \tag{2.2.2}$$

We next use the wave equation to describe the amplitude of the electric field at the screen:

$$E = E_0 \sin(kl - \omega t), \tag{2.2.3}$$

where E_0 is a proportionality constant that depends on the intensity of the incoming light, the amount transmitted by the slits, and the distance l_0. Here we are only interested in the overall shape of the interference pattern (and, in particular, its dependency on θ) so the numerical value of E_0 is unimportant. Inserting the expressions for l_1 and l_2, we have

$$E = E_0 \left[\sin\{k(l_0 - d\theta/2) - \omega t\} + \sin\{k(l_0 + d\theta/2) - \omega t\}\right] \tag{2.2.4}$$

To separate out the time dependency, we can make use of the trigonometric identity

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \tag{2.2.5}$$

If we set

$$u = kl_0 - \omega t + kd\theta/2 \tag{2.2.6}$$

and

$$v = kl_0 - \omega t - kd\theta/2 \tag{2.2.7}$$

then we see that Eq. (2.2.4) can be written as

$$E = 2E_0 \sin(kl_0 - \omega t) \cos(kd\theta/2) \tag{2.2.8}$$

Note that only the first factor depends on the time, t. We further learned in the lecture that the intensity of light on the screen $S(\theta)$ is proportional to E^2,

$$\begin{aligned} S(\theta) &\propto E(\theta)^2 & (2.2.9)\\ &= [2E_0 \sin(kl_0 - \omega t) \cos(kd\theta/2)]^2 & (2.2.10)\\ &\propto \sin^2(kl_0 - \omega t) \cos^2(kd\theta/2) & (2.2.11)\end{aligned}$$

Averaging over time, the first factor reduces to $\langle \sin^2(kl_0 - \omega t)\rangle = 1/2$. Inserting the definition of the wave number, $k \equiv 2\pi/\lambda$, we get

$$\langle S \rangle \propto \cos^2(\pi\theta d/\lambda) \tag{2.2.12}$$

We see that the diffraction pattern consists of a series of maxima of equal height and minima where the intensity of the light is zero. The first maximum occurs for $\theta = 0$, the next for $\theta = \lambda/d, 2\lambda/d, \ldots$.

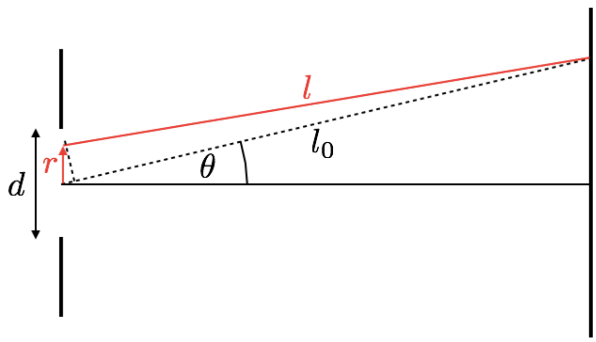
The single slit

We now turn to the case of a single, wide slit. As before, we are interested in the diffraction pattern $S(\theta)$ produced by light from an infinitely distant point source, passing through the slit, as a function of the angle θ. The situation is as illustrated in the figure below, where one should imagine looking at an "infinitely long" vertical slit from "above". We thus consider the one-dimensional problem of calculating the diffraction pattern as a function of the angle θ.

Instead of two discrete openings, the total electro(magnetic) field at a point on the screen to the right now receives contributions from a continuous range of points along the opening of width d. For a small part of the opening, of width dr and located at distance r from the centre, the contribution to E is then

$$\mathrm{d}E = E_0 \sin(kl - \omega t)\, \mathrm{d}r \tag{2.2.13}$$

To compute $S(\theta)$, we then need to sum all these contributions to get the amplitude of the electro-magnetic field, $E(\theta, t)$, square this to get the intensity $S(\theta, t)$, and finally average over time to get $\langle S(\theta)\rangle$.



a. Show that the intensity diffraction pattern for an opening of width d (apart from constant factors) is

$$\langle S(\theta)\rangle \propto \frac{\sin^2(\pi\theta d/\lambda)}{(2\pi\theta/\lambda)^2}$$

You may need the following trigonometric identity:

$$\cos(u - v) - \cos(u + v) = 2 \sin u \sin v$$

(the lecture slides include an extra factor d^2 in the denominator which normalises the pattern to unity at $\theta = 0$).

b. For given d and λ, at what θ does the first minimum occur? Compare with the Rayleigh criterion for a circular aperture.

c. According to the Rayleigh criterion, what is the angular resolution of the 35 cm telescope when observing at a wavelength of 500 nm (give the answer in arcseconds)?

d. A 1 Euro coin has a diameter of 23 mm. At what distance would such a coin have to be placed in order for its (apparent) diameter to coincide with the first dark ring in the diffraction pattern of the 35 cm telescope?

e. The closest star, proxima Centauri, is located at a distance of about 4 light years from the Sun (a light year being the distance travelled by light in a year). What would be the angular diameter of the Sun, seen from proxima Centauri? (the diameter of the Sun is about 1.4×10^6 km). Would you be able to see the Sun as an extended object from proxima Centauri with the 35 cm telescope?