ASSIGNMENTS Week 2 (F. Saueressig)

Cosmology 16/17 (NWI-NM026C)

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Exercise 1 is **the first hand-in assignment** in this course. Please submit your solution to your teaching assistant before the tutorial on **Thursday, 15th September**. Present your solution in a readable way.

Exercise 1: True or false? Lessons on how to manipulate indices (hand-in exercise)

We denote the spacetime metric by $g_{\alpha\beta}$, the inverse spacetime metric by $g^{\alpha\beta}$, and the Christoffel symbol by $\Gamma^{\lambda}{}_{\mu\nu}$. Moreover, let A, B, C be tensors of the rank indicated by their indices.

a) Carefully study the equations below. State if they are correct or not. Also explain which property has been used (perhaps in a wrong way)!

$$g_{\alpha\beta} = g_{\beta\alpha}\,, \tag{1}$$

$$g_{\alpha\beta} = g_{\beta\mu}\,, \tag{2}$$

$$g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta} = g_{\alpha\gamma}\, dx^{\alpha}\, dx^{\beta}\,, \tag{3}$$

$$\Gamma^{\alpha}{}_{\alpha\beta}\, A^{\beta} = g_{\alpha\gamma}\, A^{\alpha}\, B^{\gamma}\,, \tag{4}$$

$$\Gamma^{\sigma}{}_{\alpha\beta}\, B^{\alpha}\, C^{\beta} = \Gamma^{\sigma}{}_{\alpha\beta}\, B^{\beta}\, C^{\alpha}\,, \tag{5}$$

$$\Gamma^{\alpha}{}_{\beta\gamma}\, A^{\alpha}\, C^{\beta}\, C^{\gamma} = B^{\alpha} \tag{6}$$

$$\frac{\partial x^{\alpha}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta}\,, \tag{7}$$

$$\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} = 0\,, \tag{8}$$

$$\bar{g}_{\alpha\beta}\bar{A}^{\alpha}\bar{B}^{\beta} = g_{\alpha\beta}A^{\alpha}B^{\beta}\,, \tag{9}$$

$$\Gamma^{\alpha}{}_{\alpha\beta} - \Gamma^{\beta}{}_{\beta\beta} = 0\,. \tag{10}$$

b) Let $R_{\mu\nu}$ be a second rank tensor and let $R \equiv g^{\mu\nu} R_{\mu\nu}$ denote its trace. Explain the mistake that is done in **each step** of the following computation

$$g_{\mu\nu} R = g_{\mu\nu}\,(g^{\mu\nu}\, R_{\mu\nu}) \tag{11}$$

$$= (g_{\mu\nu}\, g^{\mu\nu})\, R_{\mu\nu} \tag{12}$$

$$= R_{\mu\nu}\,. \tag{13}$$

Exercise 2: Properties of the Christoffel Symbol and covariant derivative

Under a coordinate transformation $\bar{x}^\alpha(x^\mu)$ the metric tensor $g_{\alpha\beta}(x)$ transforms as a contravariant tensor of rank 2

$$g_{\mu\nu}(x) = \bar{g}_{\alpha\beta}(\bar{x}(x))\,\frac{\partial \bar{x}^\alpha}{\partial x^\mu}\,\frac{\partial \bar{x}^\beta}{\partial x^\nu}\,. \tag{14}$$

We also introduce the Christoffel symbols,

$$\Gamma^\alpha{}_{\mu\nu} \equiv \tfrac{1}{2}\, g^{\alpha\beta}\left[g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}\right], \tag{15}$$

which provide the connection piece in the covariant derivatives D_μ, i.e., $D_\mu v^\alpha = \partial_\mu v^\alpha + \Gamma^\alpha{}_{\mu\lambda} v^\lambda$. Show that

a) under a coordinate transformation $\bar{x}^\alpha(x^\mu)$ the Christoffel symbols transform according to

$$\bar{\Gamma}^\tau{}_{\sigma\rho} = \frac{\partial \bar{x}^\tau}{\partial x^\lambda}\,\frac{\partial x^\mu}{\partial \bar{x}^\sigma}\,\frac{\partial x^\nu}{\partial \bar{x}^\rho}\,\Gamma^\lambda{}_{\mu\nu} - \frac{\partial^2 \bar{x}^\tau}{\partial x^\mu\,\partial x^\nu}\,\frac{\partial x^\mu}{\partial \bar{x}^\sigma}\,\frac{\partial x^\nu}{\partial \bar{x}^\rho}\,. \tag{16}$$

b) Based on the general properties of the covariant derivative and the definition of the Christoffel symbol (15) show that

$$D_\alpha\, g_{\mu\nu} = 0\,. \tag{17}$$

The property eq. (17) shows that the covariant derivative is metric compatible.

Exercise 3: Computing physical distances

In units where $G = c = 1$ the Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,. \tag{18}$$

Use the "cookie recipe" provided in class to compute the physical radial distance (at equal time t) between the last stable orbit, $r = 3M$ and the horizon, $r = 2M$.