ASSIGNMENTS Week 3 (F. Saueressig)
Cosmology 16/17 (NWI-NM026C)

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Exercise 1: Weight a galaxy and discover dark matter (hand-in)

Weak gravitational lensing provides a way to measure the masses of astronomical objects without requiring assumptions about their composition or dynamical state. The goal of this exercise is to derive the central relation underlying gravitational lensing: the deflection angle is proportional to the mass of the object bending the light ray. Note that:

1. This is a standard computation in general relativity. It constitutes one of the classical tests of the theory. The derivation can be found in many textbooks (see, e.g., Hartle "Gravity – An introduction to Einstein's general relativity" Chapter 9).
2. In order to lighten the notation, one can set $G = 1$. This is done in the sequel.

The geometry outside a spherical symmetric object (star, galaxy) with mass M is described by the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) . \tag{1}$$

The metric (1) is independent of t and spherically symmetric. As a consequence any world line $x^\alpha(\lambda)$, $x^\alpha = (t, r, \theta, \phi)$, with curve parameter λ admits two conserved quantities, the energy e and the angular momentum l:

$$e = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\lambda} , \qquad l = r^2 \sin^2\theta \, \frac{d\phi}{d\lambda} . \tag{2}$$

The fact that e and l are indeed conserved quantities can be shown from the geodesic equation satisfied by the light ray and will be taken for granted at this stage (the proof is provided in the optional exercise 2).

The derivation of the deflection angle proceeds along the following lines. Consider a light ray moving in the orbital plane $\theta = \pi/2$ and coming in from $r = \infty$ (see Fig. 1).

a) Use the conserved quantities together with the fact that the propagation of the light ray has to satisfy

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \tag{3}$$

to derive a differential equation for $\frac{dr}{d\lambda}$:

$$\frac{e^2}{l^2} = \frac{1}{l^2} \left(\frac{dr}{d\lambda}\right)^2 + W_{\rm eff}(r) , \qquad W_{\rm eff}(r) = \frac{1}{r^2} \left(1 - \frac{2M}{r}\right) . \tag{4}$$

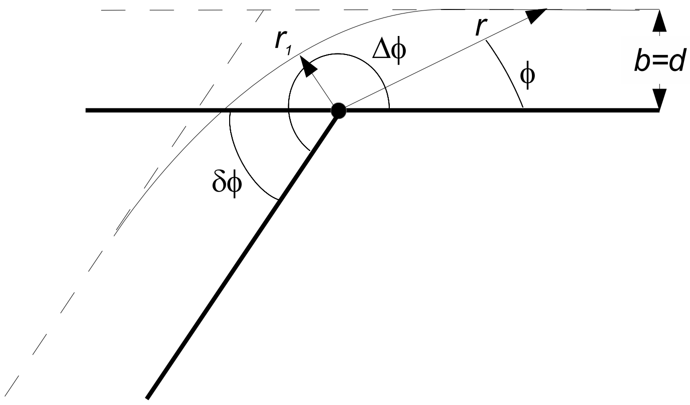

Figure 1: Quantities needed for calculating the deflection angle of light $\delta\phi$ by a spherical star. The light ray enters from the right with an impact parameter $b = d$. It approaches the gravitational source until the turning point r_1. Passing the turning point the light ray moves out to infinity, emerging deflected by an angle $\delta\phi$. The deflection angle is the total angle $\Delta\phi$ less π.

b) Define $b \equiv |l/e|$ and establish that $b = d$ is the impact parameter of the light ray, see fig. 1. This can be seen as follows. Firstly, introduce Cartesian coordinates on the equatorial plane ($\theta = \pi/2$)

$$x = r\cos\phi\,, \qquad y = r\sin\phi\,. \tag{5}$$

Denoting the coordinate distance between the incoming light ray and the x-axis by d, show that for very large $r \gg d$, $\phi \approx d/r$. Moreover, by studying the asymptotics of (3) for $r \to \infty$ show that asymptotically $dr/dt = -1$. Combine the two results and derive an expression for $d\phi/dt$.

Now evaluate b in the limit $r \gg 2M$ by substituting the conserved quantities (2). Show that in this limit

$$b = r^2\,\frac{d\phi}{dt}\,. \tag{6}$$

Combine this result with the geometric analysis to show $b = d$.

c) Combine eqs. (4) and the conserved quantities (2) to derive

$$\frac{d\phi}{dr} = \pm\frac{1}{r^2}\left[\frac{1}{b^2} - W_{\mathrm{eff}}(r)\right]^{-1/2}\,. \tag{7}$$

Integrate this equation to obtain

$$\Delta\phi = 2\int_{r_1}^{\infty}\frac{dr}{r^2}\left[\frac{1}{b^2} - W_{\mathrm{eff}}(r)\right]^{-1/2}\,. \tag{8}$$

Here r_1 denotes the turning point of the light ray, which can be determined from the condition $\frac{dr}{d\phi} = 0$. Thus bracket in (7) vanishes for $r = r_1$ by definition of the turning point.

d) Express the integral (8) in terms of the new variable $w = b/r$. Show that this leads to

$$\Delta\phi = 2\int_0^{w_1} dw \left[1 - w^2\left(1 - \frac{2M}{b}w\right)\right]^{-1/2} . \tag{9}$$

e) For our sun, the smallest value for b is approximately the radius of the sun $R_\odot = 6.96 \times 10^5$ km. The mass of the sun (in units where $G = 1$) is $M_\odot = 1.47$ km. Thus the value for $2M/b \approx 10^{-6}$ and the expression for $\Delta\phi$ may be expanded in this small quantity. Retaining the terms linear in $2M/b$, show that this results in

$$\Delta\phi \approx 2\int_0^{w_+} dw \frac{1 + (M/b)w}{[1 + (2M/b)w - w^2]^{1/2}} . \tag{10}$$

Here w_+ is the positive root of the denominator. Note that the expansion of (9) has to be done with care, expanding terms of the form $1 - \frac{2M}{b}w$ only.

f) Look up the integral (10) in an integral table. Confirm that

$$\Delta\phi \approx \pi + \frac{4M}{b} . \tag{11}$$

Compute the deflection angle $\delta\phi = \Delta\phi - \pi$, see Fig. 1.

Exercise 2: Conserved quantities in the Schwarzschild metric (optional)

The list of Christoffel symbols arising from the metric (1) is given by

$$\begin{aligned}
&\Gamma^t{}_{tr} = (M/r^2)(1 - 2M/r)^{-1} , && \Gamma^\theta{}_{r\theta} = 1/r \\
&\Gamma^r{}_{tt} = (M/r^2)(1 - 2M/r) , && \Gamma^\theta{}_{\phi\phi} = -\cos\theta\,\sin\theta \\
&\Gamma^r{}_{rr} = -(M/r^2)(1 - 2M/r)^{-1} , && \Gamma^\phi{}_{r\phi} = 1/r \\
&\Gamma^r{}_{\theta\theta} = -(r - 2M) , && \Gamma^\phi{}_{\theta\phi} = \cot\theta \\
&\Gamma^r{}_{\phi\phi} = -(r - 2M)\,\sin^2\theta .
\end{aligned} \tag{12}$$

a) Based on the Christoffel symbols (12) write down the geodesic equation for the t-component of a massive test particle moving in the Schwarzschild geometry (1).

b) Take a derivative of

$$e \equiv \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \tag{13}$$

with respect to the curve parameter λ. Show that the resulting expression is a multiple of the geodesic equation constructed in part a). Thus e is a conserved along the world line of the particle.

Remark: The fact that l is a conserved quantity can be established along the same lines, based on the ϕ-component of the geodesic equation.