ASSIGNMENTS Week 8 (F. Saueressig)

Cosmology 16/15 (NWI-NM026C)

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The hand-in of Exercise 1 is optional: if you have not submitted the hand-in in week 3 or received a "non-pass", you may make up for this by submitting your solution of Exercise 1 to your teaching assistant before the tutorial on **Wendsday, 9th November**.

Exercise 1: Stress-energy tensor for a scalar field (hand-in)

The dynamics of a scalar field (inflaton) minimally coupled to gravity is encoded in the action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} R - \tfrac{1}{2}\, g^{\mu\nu}\, \partial_\mu \phi \, \partial_\nu \phi - V(\phi) \right] . \tag{1}$$

Here $g \equiv \det g_{\mu\nu}$ denotes the determinant of the spacetime metric, $\int d^4x \sqrt{-g}$ is the physical four-volume invariant under coordinate transformations and $V(\phi)$ is the scalar potential (including a possible cosmological constant). The equations of motion are obtained via the variation principle, exploiting that classical solutions are extrema of the action. Thus we vary S with respect to the fields ϕ, $g_{\mu\nu}$ and subsequently set the variation to zero. Variations with respect to fields can be performed by utilizing the basic definitions

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \delta^4(x-y) \, , \qquad \frac{\delta g_{\mu\nu}(x)}{\delta g_{\rho\sigma}(y)} = \tfrac{1}{2} \left(\delta^\rho_\mu \delta^\sigma_\nu + \delta^\sigma_\mu \delta^\rho_\nu \right) \, \delta^4(x-y) \, . \tag{2}$$

Variation with respect to the metric furthermore obey the auxiliary identities

$$\begin{aligned} \delta g^{\mu\nu} &= - \, g^{\mu\alpha} \, g^{\nu\beta} \delta g_{\alpha\beta} \, , \\ \delta \sqrt{-g} &= \tfrac{1}{2} \sqrt{-g} \, g^{\mu\nu} \delta g_{\mu\nu} = -\tfrac{1}{2} \sqrt{-g} \, g_{\mu\nu} \delta g^{\mu\nu} \, , \\ \delta R &= R_{\mu\nu} \delta g^{\mu\nu} + \text{total covariant derivatives} \, . \end{aligned} \tag{3}$$

a) Show that the variation of (1) with respect to the scalar field results in

$$\frac{1}{\sqrt{-g(y)}} \, \partial_\mu \sqrt{-g(y)} \, g^{\mu\nu} \, \partial_\nu \, \phi - \frac{\partial V}{\partial \phi} = 0 \, . \tag{4}$$

Specialize this equation to the case where $g_{\mu\nu} = \eta_{\mu\nu}$ is the Minkowski metric and $V(\phi) = \frac{1}{2} m^2 \phi^2$ is a mass term. Verify that in this case (4) agrees with the massive Klein-Gordon equation $\left((\partial_0)^2 - (\partial_i)^2 + m^2 \right) \phi = 0$ where ∂_0 is a partial derivative with respect to the time coordinate t and ∂_i denotes a partial derivative with respect to the spatial coordinate x^i.

b) Show that the variation of (1) with respect to the metric results in Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \, T^\phi_{\mu\nu} \, . \tag{5}$$

where the stress-energy tensor

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S^{\text{matter}}}{\delta g^{\mu\nu}} \tag{6}$$

for the scalar field is given by

$$T^{\phi}_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\tfrac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + V(\phi)\right)\,. \tag{7}$$

c) Specialize the general formula (7) to the case of a homogeneous and isotropic universe where the line element is of Friedmann-Robertson-Walker form, $ds^2 = -dt^2 + a(t)^2\left(dx^2 + dy^2 + dz^2\right)$, and the scalar field $\phi = \phi(t)$ is a function of cosmic time only. Compare the result to the stress-energy tensor of a perfect fluid

$$T^{\mu\nu} = (\rho + p)\,u^\mu u^\nu + g^{\mu\nu}\,p\,. \tag{8}$$

What is the scalar's four-velocity u^μ, energy density ρ and pressure p in this case?

Exercise 2: Invariance of S^{matter} under coordinate transformations ensures $D_\mu T^{\mu\nu} = 0$

We start from a generic action S^{matter}, describing the matter content of the universe minimally coupled to the spacetime metric. Based on S^{matter}, the stress-energy tensor $T_{\mu\nu}$ entering into Einsteins equations is obtained via the variational principle via eq. (6).

a) Use the rules for variations with respect to the metric field, eqs. (2) and (3), to show that the definition (6) is compatible with

$$\delta S^{\text{matter}} = \tfrac{1}{2}\int d^4x\sqrt{-g}\,T^{\mu\nu}\,\delta g_{\mu\nu}\,. \tag{9}$$

b) Use the definition of the covariant derivative in terms of Christoffel symbols to prove

$$D_\mu T^\mu{}_\nu = \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,T^\mu{}_\nu\right) - \tfrac{1}{2}\,T^{\alpha\beta}\left(\partial_\nu\,g_{\alpha\beta}\right)\,. \tag{10}$$

c) Consider the special case where the variation (9) is induced by an infinitesimal coordinate transformation

$$(x')^\mu = x^\mu + \xi^\mu\,. \tag{11}$$

At the level of the spacetime metric this transformation induces

$$\delta g_{\mu\nu} = \xi^\alpha\,\partial_\alpha g_{\mu\nu} + (\partial_\mu\xi^\alpha)g_{\alpha\nu} + (\partial_\nu\xi^\alpha)g_{\mu\alpha}\,. \tag{12}$$

Substitute the variation (12) into (9) to establish that

$$\delta S^{\text{matter}} = -\int d^4x\sqrt{-g}\,\xi^\nu\left(\frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,T^\mu{}_\nu\right) - \tfrac{1}{2}\,T^{\alpha\beta}\left(\partial_\nu\,g_{\alpha\beta}\right)\right)\,. \tag{13}$$

Use eq. (10) to recast (13) in terms of covariant derivatives.

d) Finally, assume that S^{matter} is invariant under coordinate transformations. Explain the resulting implications for δS^{matter}. Use your results from the previous parts and argue that $D_\mu T^\mu{}_\nu = 0$ in this case.