ASSIGNMENTS Week 9 (F. Saueressig)

Cosmology 16/17 (NWI-NM026C)

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Exercise 1 is a **hand-in** assignment. Please submit your solution to your teaching assistant before the tutorial on **Wendsday, 16th November**. **Present your solution in a readable way**.

Exercise 1: Derive the Friedmann equations from Einstein's equations (hand-in)

Einstein's equations are given by

$$R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\, R = 8\pi G\, T_{\mu\nu} \tag{1}$$

with the Ricci tensor given by

$$R_{\mu\nu} \equiv \partial_\lambda\, \Gamma^\lambda{}_{\mu\nu} - \partial_\nu\, \Gamma^\lambda{}_{\lambda\mu} + \Gamma^\lambda{}_{\sigma\lambda}\Gamma^\sigma{}_{\mu\nu} - \Gamma^\lambda{}_{\sigma\nu}\Gamma^\sigma{}_{\lambda\mu} \tag{2}$$

and $R \equiv g^{\mu\nu} R_{\mu\nu}$ denoting the Ricci scalar. The matter contribution is encoded in the stress-energy tensor $T_{\mu\nu}$. The most general ansatz for a homogeneous and isotropic universe is given by the Friedmann-Robertson-Walker (FRW) metric, encoded in the line element

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right] . \tag{3}$$

Here k is a numerical constant determining weather the universe is closed ($k=+1$), flat ($k=0$), or open ($k=-1$) and all the dynamics of the model is captured by the time-dependence of the scale factor $a(t)$. The stress-energy tensor is modeled by a perfect fluid with energy density $\rho(t)$ and pressure $p(t)$ which is at rest with respect to the cosmic coordinates

$$T_\mu{}^\nu = \mathrm{diag}\left[-\rho(t)\,,\, p(t)\,,\, p(t)\,,\, p(t) \right] . \tag{4}$$

a) Evaluate Einstein's equations for the FRW metric (3) and the homogeneous and isotropic stress-energy tensor (4). Defining the Hubble parameter $H \equiv \frac{\dot{a}}{a}$ where the dot denotes a derivative with respect to cosmic time t, show that the dynamics of $a(t)$ is governed by the Friedmann equations

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, \\ \frac{\ddot{a}}{a} &= -\frac{4}{3}\pi G\left(\rho + 3p\right) . \end{aligned} \tag{5}$$

Hint: The first equation results from the tt-part of Einstein's equations while the second one stems from the spatial components of Einstein's equations, e.g., the rr-part.

b) Show that the conservation of the stress-energy tensor (4), $D_\mu T^{\mu\nu} = 0$ entails

$$\frac{d}{dt}\left[\rho(t)\, a(t)^3\right] = -p(t)\frac{d}{dt}\left[a(t)^3\right] . \tag{6}$$

c) Show that the three equations (5) and (6) are not independent. Combine the first equation in (5) and (6) to derive the second equation in (5).

Exercise 2: Singularities in the Friedman-Robertson-Walker (FRW) universe

The Big Bang singularity corresponds to a situation where the scale factor $a(t)$ goes to zero and the universe is at infinite density. Consider the flat FRW universe where $k = 0$:

a) Given that the cosmic fluid satisfies the equation of state $p(t) = w\rho(t)$, determine the condition on w so that the universe has a Big Bang singularity in the past.

b) Find the de Sitter solution of the Friedman equation by determining the scale factor as a function of time for the case that there is only vacuum energy $\rho_v > 0$ with $w = -1$. Does the model have an initial Big Bang singularity?

Exercise 3: A simplified model of our universe

Study the flat FRW universe for the case when there is no radiation, $\rho_r = 0$, but both vacuum energy and matter.

a) Defining the Hubble constant $H_0 \equiv \dot{a}(t_0)/a(t_0)$ show that the Friedman equation (evaluated today at time t_0) requires that the total energy density is $\rho_{\text{crit}} = \frac{3H_0^2}{8\pi}$.

b) Use the critical energy density to introduce the relative fractions for the matter density $\Omega_m \equiv \rho_m(t_0)/\rho_{\text{crit}}$ and $\Omega_v \equiv \rho_v(t_0)/\rho_{\text{crit}}$. Fixing the scale factor today $a(t_0) = 1$ use the energy conservation law to express the total energy density in terms of the relative fractions and the scale factor $a(t)$.

c) Use this expression to cast the Friedman equation into the form

$$\frac{1}{2\,H_0^2}\dot{a}^2 + U_{\text{eff}}(a) = 0\,. \tag{7}$$

What is the explicit form of $U_{\text{eff}}(a)$ appearing in this model? Try to find an implicit expression for $a(t)$ in terms of H_0, Ω_m and $\Omega_v = 1 - \Omega_m$.

d) How large would Ω_v have to be for the universe to be accelerating ($\ddot{a} > 0$) at the present time?

Exercise 4: Light from distant galaxies

Consider a galaxy whose light we see today at time t_0 that was emitted at time t_e. Show that the present proper distance to the galaxy (along a curve of constant t_0) is

$$d = a(t_0) \int_{t_e}^{t_0} \frac{dt}{a(t)} \,. \tag{8}$$

Exercise 5: Vacuum energy from quantum gravity?

Could the vacuum mass-energy density of the universe be a consequence of quantum gravity? While this seems intuitively natural, this explanation suffers from the great difference between the observed vacuum density ρ_v and the Planck mass density $\rho_{\rm Pl} = c^5/(\hbar G^2)$ which sets the natural scale associated with quantum gravity phenomena.

a) Show that $\rho_{\rm Pl}$ is the correct combination of $\hbar$, G and c with the dimensions of mass density.

b) Estimate the ratio $\rho_v/\rho_{\rm Pl}$. Use that $\Omega_\Lambda \equiv \rho_v/\rho_c \simeq 0.7$ where ρ_c is the critical density of the flat universe.

Remark: finding an explanation for the smallness of ρ_v is one of the greatest puzzles in theoretical physics today!