ASSIGNMENTS Week 11 (F. Saueressig)

Cosmology 16/17 (NWI-NM026C)

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Exercise 3 is a hand-in assignment. Please submit your solution to your teaching assistant before the tutorial on **Wendsday, 30th November**. Present your solution in a readable way.

Exercise 1: Heating the cosmic plasma

Typically, the cosmic plasma T cools as $T \propto a^{-1}$. When particles (e.g. primordial black holes) decay, they deposit energy in the cosmic plasma. (Note: this does not imply that the temperature of the universe actually increases, it just states that it cools slower than a^{-1}.) The entropy density s of the cosmic plasma is defined as

$$s \equiv \frac{\rho + p}{T} \, . \tag{1}$$

The entropy density scales according to $s \propto a^{-3}$ so that sa^3 is actually a conserved quantity. Use the conservation of sa^3 to compute the ratio of $(aT)^3$ at $T = 10$ GeV (the energy scale where WIMPs decouple) and its present value.

a) Note that only relativistic species contribute to the entropy density in an efficient way. Thus compute the number of effective relativistic species g_* at 10 GeV and today. (Hint: this requires some particle physics knowledge. For $T = 10$ GeV you should obtain $g_* = 86.25$. Taking into account that neutrinos decouple after e^+e^- annihilation, which lowers the temperature of the cosmic neutrino background, the effective number of relativistic degrees of freedom today is $g_* = 3.36$.).

b) Use the energy and momentum densities for relativistic particles constructed constructed in last weeks exercises to relate $(aT)^3$ at $T = 10$ GeV and today.

Exercise 2: Harmonic oscillator revisited

Consider the harmonic oscillator with constant frequency ω. Use the expansion of the position operator,

$$\hat{x} = v(t)\hat{a} + v^*(t)\hat{a}^\dagger \, , \tag{2}$$

together with the property that the vacuum is annihilated by $\hat{a}$ to show that the fluctuations of the position in the vacuum state are determined by the mode function $v(t)$:

$$\langle 0| \, |\hat{x}|^2 \, |0\rangle = |v(t)|^2 \, . \tag{3}$$

Exercise 3: Fourier-modes decouple for linearized perturbations (hand-in exercise)
Consider the linearized Einstein's equations around a general FRW background

$$\delta G_{\mu\nu} = 8\pi G\, \delta T_{\mu\nu}\,. \tag{4}$$

Owed to the homogeneity of the background the linearized equations possess a translation symmetry in the spatial coordinates, i.e., $x^i \mapsto x^i + \Delta x^i$ with Δx^i being constant is a symmetry. The Fourier-components of a general perturbation $\delta Q(t, \mathbf{x})$ are defined as

$$\delta Q(t, \mathbf{k}) = \int d^3\mathbf{x}\, \delta Q(t, \mathbf{x})\, e^{-i\mathbf{k}\mathbf{x}}\,. \tag{5}$$

Here and in the following spatial components of a vector are indicated as bold-face letters. Show that *translation invariance* implies that different Fourier modes (identified by different wave-numbers k) evolve independently.
Hint: you may use that linear equations of the form (4) may be solved by the transfer-matrix method

$$\delta Q_I(t_2, \mathbf{k}) = \sum_{J=1}^{N} \int d^3\bar{\mathbf{k}}\; T_{IJ}(t_2, t_1, \mathbf{k}, \bar{\mathbf{k}})\, \delta Q_J(t_1, \bar{\mathbf{k}})\,, \tag{6}$$

where δQ_I, $I = 1, \ldots, N$ are N independent perturbations which are evolved from initial time t_1 to the final time t_2 and the transfer matrix follows form the linearized Einstein's equations.

Exercise 4: Single-field inflation: the case $m^2\phi^2$ revisited

In the mass-driven inflation model based on the potential $V(\phi) = \frac{1}{2}m^2\phi^2$ the fluctuations forming the CMB are created at $\phi_\star = \phi_{\rm cmb}$ approximately $N_{\rm cmb} \sim 60$ e-folds before the end of inflation. Compute the scalar spectral index n_s and the tensor-to-scalar ratio r evaluated at the CMB scale.

a) Compute the slow-roll parameters $\epsilon_{\rm V}$ and $\eta_{\rm V}$ at $\phi_\star$ and express the result in terms of $N_{\rm cmb}$.

b) The massless scalar power spectrum is normalized so that $\Delta_s^2 \sim 10^{-9}$. Show that this fixes the inflaton mass to be $m \sim 10^{-6} M_{\rm Pl}$.

c) Use the slow-roll approximation to compute the scalar spectral index n_s and the scalar-to-tensor ratio r.

d) Compare the results with current cosmological data compiled by the particle physics data group (see: pdg.lbl.gov/2015/reviews/rpp2014-rev-cosmological-parameters.pdf).

Exercise 5: Working with scientific literature - a howto

Download the lecture notes on inflation: `arxiv.org/pdf/0907.5424`. Start from the action eq. (181),

$$S^{(2)} = \frac{1}{2} \int d^4x \, a^3 \, \frac{\dot{\phi}^2}{H^2} \left[\dot{\mathcal{R}}^2 - a^{-2} (\partial_i \mathcal{R})^2 \right] , \tag{7}$$

which results from the Einstein-Hilbert action supplemented by the action of a scalar field expanded to second order in the gauge-invariant curvature parameter $\mathcal{R}$.

a) Rewrite the action (7) in terms of the Mukhanov variable

$$v \equiv z \, \mathcal{R} \, , \qquad z^2 \equiv a^2 \frac{\dot{\phi}^2}{H^2} \, . \tag{8}$$

Show that the result is given by

$$S^{(2)} = \frac{1}{2} \int d\tau d^3x \left[(v')^2 - (\partial_i v)^2 + \frac{z''}{z} v^2 \right] . \tag{9}$$

Here τ is the conformal time variable and the prime denotes a derivative with respect to τ. Compare (9) to eq. (183) in the lecture notes.

b) Verify that the sign mistake in eq. (183) "does not propagate", i.e., check that

$$v_k'' + \left(k^2 - \frac{z''}{z} \right) v_k = 0 \tag{10}$$

is indeed the correct equation of motion for the Fourier mode.

Remark: This exercise is a very valuable lesson in good scientific conduct: if you copy equations from literature sources, make sure that, firstly, you understand the conventions and, secondly, verify that the equations which you are using are correct.