Werkcollege, Cosmology 2016/2017, Week 13

These are the exercises and hand-in assignment for the 13th week of the course *Cosmology*. The hand-in assignment for this week is **Problem 13.5** below.

13.1 Gravitational microlensing

In 1986, Bohdan Paczyński suggested that dark matter in the form of massive compact halo objects (MACHOs) would be detectable due to gravitational lensing of distant stars. Recall the expression for the angular radius of the Einstein ring:

$$\theta_E^2 = \frac{4GM}{c^2}\left(\frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}D_{\mathrm{L}}}\right) \tag{13.1.1}$$

where M is the mass of the lensing object, D_{LS} is the distance from the lens to the source, and D_{S} and D_{L} are the distances from the observer to the source and lens, respectively.

- A requirement for significant amplification of the source is that it is smaller than the Einstein radius of the lens. Calculate the minimum detectable lensing mass, assuming that the lensing objects are at a typical distance of 10 kpc and that the background stars are solar-type stars in the Large Magellanic Cloud at a distance of 50 kpc.
- Show that, for a population of lenses with a uniform spatial distribution of (mass) density ρ extending all the way to the source population, the optical depth is

$$\tau = \left(\frac{2\pi}{3}\right)\left(\frac{G\rho}{c^2}\right)D_S^2$$

- Show that, if the population of lenses forms a self-gravitating system extending all the way to the source population, then the optical depth depends only on the velocity dispersion σ of the system:

$$\tau \approx \sigma^2/c^2$$

You will need to the following quantities:
1 $R_\odot = 7 \times 10^8$ m
1 pc $= 3.08 \times 10^{16}$ m
1 $M_\odot = 2 \times 10^{30}$ kg
$G = 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}

13.2 The flatness problem

In this assignment we explore the evolution of the density parameter for matter, Ω_M, with redshift/scale factor. According to the CMB measurements by the Planck satellite, the *current* value is $\Omega_{M,0} = 0.31$ while the dark energy density parameter is $\Omega_{\Lambda,0} = 0.69$, making the Universe exactly flat with $\Omega = 1.0$.

- Recall that the critical density, at any epoch, is defined as

$$\rho_c = 3H^2/8\pi G \tag{13.2.1}$$

  and the time derivative of the scale factor is given by the Friedman equation,

$$\dot{a} = H_0 \left[\Omega_{M,0}(1/a - 1) + \Omega_{\Lambda,0}(a^2 - 1) + 1\right]^{1/2} \tag{13.2.2}$$

  Now show that regardless of the present-day values of $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$, the Universe approached an Einstein-de Sitter Universe with $\Omega_M = 1$ at high redshift.

- If the matter density is currently $\Omega_{M,0} = 0.3$, then how much did it deviate from unity at $z = 1000$?

This is the *flatness* problem: Why is the current value of Ω_M close to, but not exactly unity? It requires an exceedingly accurate degree of fine-tuning to produce the tiny departure from $\Omega_M = 1$ at high redshifts that result in a present-day Universe whose density parameter is neither very different from, nor exactly equal to unity.

From a practical perspective, however, it is very convenient that the Universe behaved as an Einstein-de Sitter Universe until relatively recently (in cosmological terms). In terms of structure formation, the regime of linear grown occurred under conditions where the density was very close to the critical value and the Ω_Λ term negligible. At later epochs this is no longer the case, but since the non-linear regime has to be treated numerically in any case the departures from the Einstein-de Sitter Universe do not represent a very serious extra complication.

13.3 Parametric solutions to Friedman's equation

Show that the parametric solutions

$$a(\theta) = \frac{\Omega_{M,0}}{2(\Omega_{M,0} - 1)}(1 - \cos\theta) \tag{13.3.1}$$

$$t(\theta) = \frac{\Omega_{M,0}}{2H_0(\Omega_{M,0} - 1)^{3/2}}(\theta - \sin\theta) \tag{13.3.2}$$

satisfy the Friedman equation (13.2.2) for $\Omega_{\Lambda,0} = 0$ and $\Omega_{M,0} > 1$.

13.4 Tophat model

In the "tophat" model for the evolution of overdensities we consider each overdensity as a "mini-Universe" with density $\Omega_0' > 1$ that evolves in a "background Universe" with $\Omega_0 = 1$. Hence, the scale factor of the background Universe evolves as

$$a = \left(\frac{3H_0 t}{2}\right)^{2/3} \tag{13.4.1}$$

while the overdensities evolve according to the parametric solutions in Problem 13.3.

- Show that the density contrast of an overdensity, once it has reached virial equilibrium, is

$$\rho_{\rm vir}/\rho_0 \approx 150 \tag{13.4.2}$$

13.5 The Press-Schechter mass function

In the lecture we saw how a few basic assumptions lead to a simple analytical formula that provides a remarkably good description of the mass function of bound structures in the Universe:

1. The Universe "initially" (i.e. shortly after the epoch of reionization) consists of particles that are distributed randomly. The variance on the mass within a given volume V is, in this case,

   $$\Sigma_V^2 = \sigma^2 V \tag{13.5.1}$$

   where σ^2 is the variance per unit volume.

2. The distribution of overdensities $P(\Delta, V)$ is Gaussian with variance given by (13.5.1)

3. The fluctuations are initially small and grow linearly until they reach a critical value, $\Sigma_{\rm crit}$, at which point they immediately collapse and virialize.

These assumptions lead to a mass function for bound fluctuations of the form

$$\frac{\mathrm{d}N}{\mathrm{d}M} \propto M^{-3/2} \exp(-M/M^\star) \tag{13.5.2}$$

where $M^\star \propto a^2$ for scale factor a.

A more general result may be obtained by relaxing the assumption (13.5.1).

- Suppose that the variance follows a relation of the form

  $$\Sigma_V^2 = \sigma^2 V^{2\alpha} \tag{13.5.3}$$

  Then, following the same reasoning that led to (13.5.2) (see the lecture viewgraphs), show that the more general mass function has the form

  $$\frac{\mathrm{d}N}{\mathrm{d}M} \propto M^{-1-\alpha} \exp\left(-\left[\frac{M}{M^\star}\right]^{2(1-\alpha)}\right) \tag{13.5.4}$$

  with

  $$M^\star \propto a^{1/(1-\alpha)} \tag{13.5.5}$$

  The relation

  $$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{erfc}\left(a\xi^b\right) = -\frac{2ab\exp\left(-a^2\xi^{2b}\right)\xi^{b-1}}{\sqrt{\pi}} \tag{13.5.6}$$

  might be useful.
