Werkcollege, Cosmology 2016/2017, Week 14

These are the exercises and hand-in assignment for the 14th week of the course *Cosmology*. The hand-in assignment for this week is **Problem 14.4** below.

14.1 Decaying potentials

We have seen in earlier lectures that small density perturbations in a Universe dominated by pressure-less dark matter grow linearly with the scale factor, i.e.,

$$\frac{\delta\rho}{\rho} \propto a \tag{14.1.1}$$

Here we examine the evolution of perturbations of the underlying potential, Ψ. Let us assume for simplicity that the perturbations are spherically symmetric.

- Suppose that a test particle is located at the outer "boundary" of a perturbation with co-moving radius r. Use the classical definition of the gravitational potential to show that, in the linear regime, the perturbation of the potential $\delta\Psi$ remains constant as the scale factor increases.
- Also show that, if the perturbations grow more slowly than a, the perturbation of the potential will decay as the scale factor increases.

14.2 Newtonian equivalence of metric perturbations

(From Dodelson, Exercise 3, Chapter 4)

The metric for a particle travelling in the presence of a gravitational field is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{00} = -2\phi$ where ϕ is the Newtonian gravitational potential; $h_{i0} = 0$ and $h_{ij} = -2\phi\delta_{ij}$:

$$g_{\mu\nu} = \begin{pmatrix} -1-2\phi & 0 & 0 & 0 \\ 0 & 1-2\phi & 0 & 0 \\ 0 & 0 & 1-2\phi & 0 \\ 0 & 0 & 0 & 1-2\phi \end{pmatrix} \tag{14.2.1}$$

- Show that $\Gamma^i{}_{00} = \delta^{ij}\partial\phi/\partial x^j$
- Show that the space components of the geodesic equation lead to $\mathrm{d}^2x^i/\mathrm{d}t^2 = -\delta^{ij}\mathrm{d}\phi/\mathrm{d}x^j$ in agreement with Newtonian theory. Use the fact that the particle is non-relativistic so $P^0 \gg P^i$.

14.3 Four-momentum of photons in perturbed FRW metric

We adopt the perturbed version of the FRW metric as follows:

$$g_{\mu\nu} = \begin{pmatrix} -1-2\Psi(x,t) & 0 & 0 & 0 \\ 0 & a^2[1+2\Phi(x,t)] & 0 & 0 \\ 0 & 0 & a^2[1+2\Phi(x,t)] & 0 \\ 0 & 0 & 0 & a^2[1+2\Phi(x,t)] \end{pmatrix} \tag{14.3.1}$$

In the lecture we found that, to first order, the 0th component of the energy-momentum four-vector can be written as

$$P^0 \simeq p(1-\Psi) \tag{14.3.2}$$

where

$$p \equiv g_{ij}P^iP^j \tag{14.3.3}$$

- Now show that the other components of the momentum four-vector can be written as

$$P^i \simeq p\hat{p}^i \frac{1-\Phi}{a} \tag{14.3.4}$$

  where $\hat{p}$ is the unit vector parallel to p.

14.4 The momentum time derivative

We have expanded the left-hand side of the Boltzmann equation in terms of the partial derivatives with respect to t, x and p as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \cdot \frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial f}{\partial \hat{p}^i} \cdot \frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t} \tag{14.4.1}$$

Using the definitions of p and $\hat{p}$, and keeping only first-order terms, we saw how this reduces to

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a} \cdot \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} \tag{14.4.2}$$

The momentum term is non-trivial and requires a bit more work. So let's get started! First, we use the 0th component of the geodesic equation:

$$\frac{\mathrm{d}^2 x^0}{\mathrm{d}\lambda^2} = -\Gamma^0{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} \tag{14.4.3}$$

- Show that, in first instance, Eq. (14.4.3) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p(1-\Psi)\right] = -\Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+\Psi) \tag{14.4.4}$$

  (i.e., Eq. 4.23 in Dodelson's book). *Hint:* as usual, keep only first order terms (linear in Ψ)!

- Next, expand out the time derivative on the left-hand side and show that this leads to

$$\frac{\mathrm{d}p}{\mathrm{d}t}(1-\Psi) = p\frac{\mathrm{d}\Psi}{\mathrm{d}t} - \Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+\Psi) \tag{14.4.5}$$

  (i.e. Eq. 4.24 in the book)

- Now, multiply by $(1+\Psi)$ to find Eq. (4.25):

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p\left(\frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right) - \Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+2\Psi) \tag{14.4.6}$$

- Finally, evaluate the Christoffel symbol and show that

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -p\left(H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right) \tag{14.4.7}$$

  Hint: See p. 91–92 in Dodelson's book.

We have now finished manipulating the left-hand side of the Boltzmann equation for photons:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a} \cdot \frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left(H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right) \tag{14.4.8}$$

14.5 First order terms of the Boltzmann equation for photons

- Demonstrate that the *first-order* terms in the left-hand side of the Boltzmann equation for photons (Equation (4.40) in Dodelson's book),

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_1 = -p\frac{\partial}{\partial t}\left(\frac{\partial f^{(0)}}{\partial p}\Theta\right) - p\frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}\left(\frac{\partial f^{(0)}}{\partial p}\right) + Hp\Theta\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right) - p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] \quad (14.5.1)$$

follow from expression (14.4.8), combined with the perturbed expansion of the photon distribution,

$$f = f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta \quad (14.5.2)$$

- The next equation in the book, (4.41), says that the first of these terms can be written as

$$-p\frac{\partial}{\partial t}\left(\frac{\partial f^{(0)}}{\partial p}\Theta\right) = -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial t} - p\Theta\frac{\mathrm{d}T}{\mathrm{d}t}\frac{\partial^2 f^{(0)}}{\partial T\partial p} \quad (14.5.3)$$

$$= -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial t} + p\Theta\frac{\mathrm{d}T/\mathrm{d}t}{T}\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right) \quad (14.5.4)$$

Show that the second term in Eq. (14.5.4) does indeed cancel the third term in Eq. (14.5.1) so that the first-order terms of the left-hand side of the Boltzmann equation for photons become

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_1 = -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] \quad (14.5.5)$$

14.6 Exercise 5, Chapter 4

Suppose we started chapter 4 by writing

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = C' \quad (14.6.1)$$

Change from this form to the one in Eq. (4.1) (with $\mathrm{d}f/\mathrm{d}t$ *on the left). How is the collision term here, C' related to C in Eq. (4.1)? Argue that the first-order perturbations in the factor relating the two collision terms can be dropped since the collision terms themselves are first-order.*

14.7 The Einstein tensor in the perturbed FRW metric

To calculate the perturbations of the metric, Ψ and Φ, given the inhomogeneities in the distribution of matter and radiation, we need Einstein's field equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{14.7.1}$$

with the Einstein tensor given by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} \tag{14.7.2}$$

Specifically, we choose the $(0,0)$ component, with

$$G^0{}_0 = g^{0i} G_{i0} = (-1 + 2\Psi) R_{00} - \frac{\mathcal{R}}{2} \tag{14.7.3}$$

for Ricci tensor

$$R_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} + \Gamma^\alpha{}_{\beta\alpha}\Gamma^\beta{}_{\mu\nu} - \Gamma^\alpha{}_{\beta\nu}\Gamma^\beta{}_{\mu\alpha} \tag{14.7.4}$$

and Ricci scalar $\mathcal{R} = g^{\mu\nu} R_{\mu\nu}$.

To calculate $\mathcal{R}$, we need all elements of $R_{\mu\nu}$ and thus the complete set of Christoffel symbols. Here, we calculate a few of them.

- Show the following relations (as usual, to first order in the perturbations of the metric):

$$\Gamma^0{}_{00} \simeq \Psi_{,0} \tag{14.7.5}$$

$$\Gamma^0{}_{i0} \simeq i k_i \tilde{\Psi} \tag{14.7.6}$$

$$\Gamma^0{}_{ij} \simeq \delta_{ij} a^2 \left[H + 2H(\Phi - \Psi) + \Phi_{,0} \right] \tag{14.7.7}$$

  where the tilde denotes the transformation to Fourier space.