Werkcollege, Cosmology 2017/2016, Week 15

These are the exercises for the 15th week of the course *Cosmology*. The hand-in assignment for this week is **Problem 15.3** below.

15.1 Momenta of the photon perturbations

Show that

$$\int_{-1}^{1} \mathrm{d}\mu\mu^2\Theta(\mu) = \frac{2}{3}\Theta_0 - \frac{4}{3}\Theta_2 \tag{15.1.1}$$

15.2 From inhomogeneities to anisotropies (I)

In this exercise we fill in some of the details in the calculation of *anisotropies* in the observed temperature distribution on the sky from the *inhomogeneities* around recombination.

We start, once again, from the Boltzmann equation for photons:

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu\mathbf{v}_b\right] \tag{15.2.1}$$

We are now, of course, interested in the high-order moments Θ_l that correspond to the (small) variations in the CMB temperature observed *today*, from our viewpoint at $\eta = \eta_0$. We start by subtracting $\dot{\tau}\Theta$ from both sides:

$$\begin{aligned}
\dot{\Theta} + ik\mu\Theta - \dot{\tau}\Theta &= -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu\mathbf{v}_b\right] - \dot{\tau}\Theta & (15.2.2)\\
\dot{\Theta} + (ik\mu - \dot{\tau})\Theta &= -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 + \mu\mathbf{v}_b\right] & (15.2.3)
\end{aligned}$$

- Verify that the left-hand side can be rewritten as

$$e^{-ik\mu\eta+\tau}\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\Theta e^{ik\mu\eta-\tau}\right] = \dot{\Theta} + (ik\mu - \dot{\tau})\Theta \tag{15.2.4}$$

We define the right-hand side as the *source function* (borrowing terminology from the theory of radiative transfer in stellar atmospheres, which shares many aspects with this calculation),

$$\tilde{S} \equiv -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 + \mu\mathbf{v}_b\right] \tag{15.2.5}$$

- Then show that the perturbations at conformal time η_0 are related to those at η_{init} as

$$\Theta(\eta_0) = \Theta(\eta_{\mathrm{init}})e^{ik\mu\eta_{\mathrm{init}}-\tau}e^{-ik\mu\eta_0+\tau} + e^{-ik\mu\eta_0+\tau}\int_{\eta_{\mathrm{init}}}^{\eta_0}\mathrm{d}\eta\tilde{S}\,e^{ik\mu\eta-\tau} \tag{15.2.6}$$

  and, if η_0 is today and η_{init} is long before recombination

$$\Theta(\eta_0) \simeq \int_{\eta_{\mathrm{init}}}^{\eta_0}\mathrm{d}\eta\tilde{S}\,e^{ik\mu(\eta-\eta_0)-\tau(\eta)} \tag{15.2.7}$$

15.3 From inhomogeneities to anisotropies (II)

We now need to calculate the multipole moments, defined as

$$\Theta_l \equiv \frac{1}{(-i)^l}\int_{-1}^{1}\frac{d\mu}{2}\mathcal{P}_l(\mu)\Theta(\mu,\eta_0) \tag{15.3.1}$$

with $\Theta(\mu,\eta_0)$ given by

$$\Theta(\mu,\eta_0) = \int_0^{\eta_0} d\eta \tilde{S}\, e^{ik\mu(\eta-\eta_0)-\tau(\eta)} \tag{15.3.2}$$

If $\tilde{S}$ did not depend on μ, this would be easy since

$$\int_{-1}^{1}\frac{d\mu}{2}\mathcal{P}_l(\mu)e^{ik\mu(\eta-\eta_0)} = \frac{1}{(-i)^l}j_l[k(\eta-\eta_0)] \tag{15.3.3}$$

where j_l is the spherical Bessel function of order l. So let us split $\tilde{S}$ into two parts,

$$\tilde{S}_1 \equiv -\dot{\Phi} - \dot{\tau}\Theta_0 \tag{15.3.4}$$

$$\tilde{S}_2 \equiv -\mu(ik\Psi + \dot{\tau}v_b) \tag{15.3.5}$$

where $\tilde{S}_2$ depends on μ and $\tilde{S}_1$ does not.

- Evaluate the part of Θ_l involving $\tilde{S}_1$ (call it $\Theta_{l,1}$) and show that

$$\Theta_{l,1} = (-1)^l\int_0^{\eta_0} d\eta e^{-\tau}\left(-\dot{\Phi} - \dot{\tau}\Theta_0\right) j_l[k(\eta-\eta_0)] \tag{15.3.6}$$

Next, we make use of the fact that $\tilde{S}_2$ appears multiplied by $e^{ik\mu(\eta-\eta_0)}$.

- Demonstrate that μ, when appearing in this context, can be replaced by

$$\mu \to \frac{1}{ik}\frac{d}{d\eta} \tag{15.3.7}$$

- Show that the integral involving $\tilde{S}_2$ evaluates to

$$\int_0^{\eta_0} d\eta\tilde{S}_2 e^{ik\mu(\eta-\eta_0)-\tau(\eta)} = \text{Const} - \int_0^{\eta_0} d\eta\, e^{ik\mu(\eta-\eta_0)}\frac{d}{d\eta}\left[e^{-\tau(\eta)}\left(-\Psi + \frac{i\dot{\tau}v_b}{k}\right)\right] \tag{15.3.8}$$

  where Const is independent of μ (so irrelevant when computing the Θ_l). Use that $\tau(0) \gg 1$ so that $e^{-\tau(0)} \simeq 0$. In case you forgot, here is the formula for integrating by parts:

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx \tag{15.3.9}$$

- Then show that the contribution of $\tilde{S}_2$ to the multipole moments is

$$\Theta_{l,2} = (-1)^l\int_0^{\eta_0} d\eta\frac{d}{d\eta}\left[e^{-\tau(\eta)}\left(\Psi - \frac{i\dot{\tau}v_b}{k}\right)\right] j_l[k(\eta-\eta_0)] \tag{15.3.10}$$

- Finally, introducing the visibility function $g(\eta) = -\dot{\tau}e^{-\tau}$, verify that the following two forms of the source function S are equivalent:

$$S(k,\eta) = e^{-\tau}(-\dot{\Phi} - \dot{\tau}\Theta_0) + \frac{\mathrm{d}}{\mathrm{d}\eta}\left[e^{-\tau(\eta)}\left(\Psi - \frac{i\dot{\tau}v_b}{k}\right)\right] \tag{15.3.11}$$

$$= g(\eta)[\Theta_0 + \Psi] + \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{iv_b g(\eta)}{k}\right) + e^{-\tau}\left[\dot{\Psi}(k,\eta) - \dot{\Phi}(k,\eta)\right] \tag{15.3.12}$$

The latter form shows more clearly that the observed CMB anisotropies contain terms of three types: 1) The monopole of the temperature perturbations combined with metric perturbations around the recombination, 2) The bulk velocity (which is coupled to the temperature dipole), also around recombination, 3) Temporal variations in the metric perturbations (i.e., the potential) along the entire line-of-sight.