2. An introduction to tensor calculus

goal of this lecture:

- provide mathematical background underlying the computation of particle trajectories or the metric field generated by gravitational sources. If you are at ease with these concepts getting "the physics behind the formulas" will be a lot easier and a lot more fun.

The beginning of modern cosmology is marked by two fundamental ideas:

1) Einstein's equivalence principle

Gravitational and inertial mass are identical, or, equivalently gravity acts on all bodies in the same way, irrespectively of their constitution.

Consequence:

- gravity can be formulated as a field theory. Instead of being related to specific properties of individual objects it can be formulated as a property of spacetime. The information about the gravitational interaction is carried by the dynamical metric field

$$g_{\alpha\beta}(x)$$

2. Physics is independent of the choice of coordinates

- requires the formulation of physics laws in terms of objects which transform covariantly under a change of coordinates $x^\alpha \to \bar{x}^\alpha(x)$

  i.e. physics laws are formulated in terms of tensors

- one can proof that there is always a choice of coordinates ξ^α such that at a fixed point x_p:

$$g_{\alpha\beta}(x_p) = \eta_{\alpha\beta} \qquad ; \qquad \partial_\mu g_{\alpha\beta}\big|_{x_p} = 0$$

  In these coordinates the laws of non-gravitational physics agree with the laws of special relativity.

  The coordinates ξ^α realize "local inertial frames".

Remarks on conventions:

- typically I will work in "god-given" units

  $\hbar = c = k_B = 1$

- The metric tensor uses "mostly plus" conventions (following Dodelsons book)

  Thus our Minkowski metric is

$$\eta_{\alpha\beta} = \mathrm{diag}(-1, +1, +1, +1)$$

- I use Einsteins sum conventions
  - a (spacetime) index α appearing twice is summed over

Measuring distances with a spacetime metric

Def : proper time τ

In special relativity (SR) the "ticking of a clock" depends on the motion of the particle carrying it (time - dilatation). The time measured by a clock carried by a particle is called the particles "proper time" τ. In SR the infinitesimal proper time intervals are related to the spacetime metric $\eta_{\alpha\beta}$: ($c = 1$)

$$- d\tau^2 = \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

x^α : 4 - vector constructed from spacetime coordinates,

e.g. $x^\alpha = (t, x, y, z)$

dx^α : infinitesimal coordinate interval

a particle then follows a world line $x^\alpha(\tau)$:

- $x^\alpha(\tau) \equiv (t(\tau), x(\tau), y(\tau), z(\tau))$

specifies the position of the particle in spacetime at each instant in proper time τ.

- given a world line $x^\alpha(\tau)$ the 4 - velocity of the particle is defined as

$$u^\alpha(\tau) \equiv \frac{dx^\alpha(\tau)}{d\tau}$$

- tangent vector to the world line

Owed to the definition of proper time:

$$\eta_{\alpha\beta}\, u^{\alpha} u^{\beta} = \eta_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau} = -1$$

$\Rightarrow$ normalizes the length of the 4-velocity.

<u>From special to general relativity:</u>

The metric tensor $\eta_{\alpha\beta}$ of special relativity is generalized to the metric field $g_{\alpha\beta}(x)$ which may depend on the spacetime coordinates $x^{\alpha} = (t, x, y, z)$.

Distances are measured by the generalized line element

$$ds^2 = g_{\alpha\beta}(x)\, dx^{\alpha} dx^{\beta}$$

Proper time:

$$-d\tau^2 = g_{\alpha\beta}(x)\, dx^{\alpha} dx^{\beta}$$

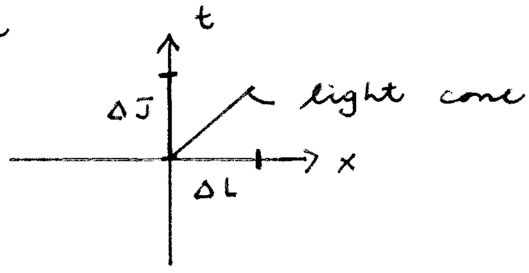
length of a ruler (spatial distances)

$$dL^2 = g_{\alpha\beta}(x)\, dx^{\alpha} dx^{\beta}$$

distance between successive positions of a light ray:

$$g_{\alpha\beta}(x)\, dx^{\alpha} dx^{\beta} = 0$$

SR - example



Computing physical distances: the cookie recipe

question: how to evaluate:

$$L = \int dL = \int \sqrt{g_{\alpha\beta}\, dx^\alpha\, dx^\beta} \qquad (*)$$

note: proper time integrals are done along the same lines.

All steps are illustrated by the specific example of the last lecture

- Consider the spacetime metric of a flat FRW universe

$$ds^2 = -dt^2 + a^2(t)\,(dx^2 + dy^2 + dz^2)$$

$$\Rightarrow \quad g_{\alpha\beta} = \mathrm{diag}\,(-1,\, a^2(t),\, a^2(t),\, a^2(t))$$

goal: compute the physical distance between

- us: located at $\vec{x} = 0$ (origin of spatial slices)
- a galaxy at $\vec{x}_G = (x_p, y_p, z_p)$

at a fixed instant of time $t = t_p$.

Step 1:

Parameterize the object / ruler / curve you want to measure. Since $(*)$ is independent of the choice of parameterization any parameterization will work.

Example:

- We have to find a parameterization $x^\alpha(u)$ of a curve connecting the origin to the position of the galaxy:

Solution :

$$x^\alpha(u) = \begin{pmatrix} t_p \\ u\, x_p \\ u\, y_p \\ u\, z_p \end{pmatrix} \qquad u \in [0, 1]$$

2) Compute the tangent vector to the curve. $\dfrac{dx^\alpha(u)}{du}$.

Example :

$$\frac{dx^\alpha(u)}{du} = \begin{pmatrix} 0 \\ x_p \\ y_p \\ z_p \end{pmatrix}$$

3) Substitute the parametrized curve into (*)

Convert the line integral into an integral w.r.t. the curve parameter u :

$$L = \int dL = \int du \sqrt{g_{\mu\nu}(x^\alpha(u)) \frac{dx^\mu}{du} \frac{dx^\nu}{du}}$$

$\uparrow$

metric evaluated on the curve

Example :

$$L = \int_0^1 du \sqrt{(-1 \cdot 0^2 + a^2(t_p)\, x_p^2 + a^2(t_p)\, y_p^2 + a^2(t_p)\, z_p^2}$$

$$= a(t_p) \sqrt{x_p^2 + y_p^2 + z_p^2} \int_0^1 du$$

4) Carry out the integral w.r.t. the curve parameter

$$L = a(t_p) \sqrt{x_p^2 + y_p^2 + z_p^2}$$

remarks :

- identifying $L \equiv d_{phys}$ this is the formula of last lecture. In particular we see that a change in the scale factor can affect the physical distance even in the case that the coordinate distance $\sqrt{x_p^2 + y_p^2 + z_p^2}$ is constant.

Coordinate transformations and tensors

Distances in spacetime are measured with the metric tensor

$$g_{\alpha\beta}(x)$$

- note $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)$ is symmetric by definition

Consider the change to a new set of coordinates $\bar{x}^{\mu}(x^{\alpha})$ with associated Jacobian J (typically a 4×4 matrix)

$$J \equiv \frac{\partial \bar{x}^{\rho}}{\partial x^{\mu}}$$

J satisfying the chain rule :

$$\frac{\partial \bar{x}^{\rho}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\rho}} = \frac{\partial x^{\nu}}{\partial x^{\mu}} = \delta^{\nu}_{\mu} \qquad \delta^{\nu}_{\mu} = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases}$$

The metric field $g_{\alpha\beta}(x)$ is a prototypical tensor. Under a change of coordinates $\bar{x}^{\mu}(x^{\alpha})$:

$$\bar{g}_{\rho\sigma}(\bar{x}) = g_{\mu\nu} \frac{\partial x^{\mu}}{\partial \bar{x}^{\rho}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\sigma}}$$

Transformation properties of infinitesimal physical distances

- Coordinate 1-forms transform via chain-rule:

$$d\bar{x}^{\rho} = \frac{\partial \bar{x}^{\rho}}{\partial x^{\mu}}\, dx^{\mu}$$

The line-element is invariant under a change of coordinates:

$$\bar{g}_{\rho\sigma}(\bar{x})\, d\bar{x}^{\rho}\, d\bar{x}^{\sigma}$$

$$\delta^{\nu}_{\beta}$$

$$= g_{\mu\nu}\, \frac{\partial x^{\mu}}{\partial \bar{x}^{\rho}}\, \frac{\partial x^{\nu}}{\partial \bar{x}^{\sigma}} \left(\frac{\partial \bar{x}^{\rho}}{\partial x^{\alpha}}\, dx^{\alpha} \right) \left(\frac{\partial \bar{x}^{\sigma}}{\partial x^{\beta}}\, dx^{\beta} \right)$$

$$\delta^{\mu}_{\alpha}$$

$$= g_{\mu\nu}\, dx^{\mu}\, dx^{\nu}$$

Example: $\mathbb{R}^2$

- cartesian coordinates:

$$ds^2 = dx^2 + dy^2 \qquad\qquad g_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- change to cylindrical coordinates

$$x = \bar{r}\cos\bar{\varphi} \quad , \quad y = \bar{r}\sin\bar{\varphi}$$

- Jacobian is a 2×2 matrix:

$$x^{\mu} = (x, y) \quad , \quad \bar{x}^{\mu} = (\bar{r}, \bar{\varphi})$$

$$\frac{\partial x^{\mu}}{\partial \bar{x}^{\rho}} = \begin{bmatrix} \cos\bar{\varphi} & \sin\bar{\varphi} \\ -\bar{r}\sin\bar{\varphi} & \bar{r}\cos\bar{\varphi} \end{bmatrix}$$

- Transformed metric tensor:

$$\bar{g}_{\mu\nu}(\bar{x}) = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu}$$

$$= \begin{bmatrix} \cos\bar{\varphi} & \sin\bar{\varphi} \\ -\bar{r}\sin\bar{\varphi} & \bar{r}\cos\bar{\varphi} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \cos\bar{\varphi} & -\bar{r}\sin\bar{\varphi} \\ \sin\bar{\varphi} & \bar{r}\cos\bar{\varphi} \end{bmatrix}}_{J^T}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \bar{r}^2 \end{bmatrix}$$

note: one has to be careful translating Einstein's sum convention in conventional matrix multiplication. The def. of J implies that $\frac{\partial x^\alpha}{\partial \bar{x}^\mu} g_{\alpha\beta} \frac{\partial x^\beta}{\partial \bar{x}^\nu} = J \cdot g \cdot J^T$ in terms of matrix multiplication. (Recall in matrix multiplication the row-index of the left matrix is contracted with the column index of the right matrix).

This gives the transformed line element in $\bar{x}$-coordinates:

$$ds^2 = d\bar{r}^2 + \bar{r}^2 d\bar{\varphi}^2$$

The metric is a prototypical example of a covariant tensor of rank 2 ($\equiv$ 2 lower indices)

The coordinate one-form dx^α is a contravariant tensor of rank 1 ($\equiv$ 1 upper index)

<u>Definition : Tensor</u>

The statement that a field with m upper and n lower indices is a tensor with respect to coordinate transformations implies that:

- each lower (covariant) index transforms with a factor $\dfrac{\partial x^\mu}{\partial \bar{x}^\alpha}$
- each upper (contravariant) index transforms with a factor $\dfrac{\partial \bar{x}^\mu}{\partial x^\alpha}$

Example:

Transformation of a mixed tensor $T_\mu{}^\nu$ of rank (1,1):

$$\bar{T}_\mu{}^\nu(\bar{x}) = T_\alpha{}^\beta(x(\bar{x}))\ \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\ \frac{\partial \bar{x}^\nu}{\partial x^\beta}$$

remarks:

- the relevance for tensors in physics comes from the fact that they allow to write equations with a well-defined transformation behavior under a change of coordinates or other symmetries (e.g. gauge symmetries).

Examples of tensors frequently appearing in physics:

1) The Kronecker - symbol δ^μ_ν is a tensor of rank (1,1):

$$\delta^\mu_\nu \frac{\partial \bar{x}^\rho}{\partial x^\mu} \frac{\partial x^\nu}{\partial \bar{x}^\sigma} = \frac{\partial \bar{x}^\rho}{\partial x^\mu} \frac{\partial x^\mu}{\partial \bar{x}^\sigma} = \frac{\partial \bar{x}^\rho}{\partial \bar{x}^\sigma} = \delta^\rho_\sigma$$

2) The inverse metric $g^{\alpha\beta}(x)$ defined by:

$$g^{\alpha\mu}(x) \; g_{\mu\beta}(x) = \delta^\alpha_\beta$$

transforms as a contravariant tensor of rank 2:

$$\begin{aligned}
\delta^\mu_\nu &\overset{?!}{=} \bar{g}^{\mu\lambda}(\bar{x}) \; \bar{g}_{\lambda\nu}(\bar{x}) \\
&\overset{!}{=} \Big(g^{\rho\sigma} \frac{\partial \bar{x}^\mu}{\partial x^\rho} \underbrace{\frac{\partial \bar{x}^\lambda}{\partial x^\sigma} \Big)\Big(g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\lambda}}_{\delta_\sigma{}^\alpha} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \Big) \\
&= \underbrace{g^{\rho\alpha} \; g_{\alpha\beta}}_{\delta^\rho{}_\beta} \frac{\partial \bar{x}^\mu}{\partial x^\rho} \frac{\partial x^\beta}{\partial \bar{x}^\nu} = \delta^\mu_\nu
\end{aligned}$$

□

3) a scalar field $\phi(x)$ is a rank 0 tensor:

$$\phi(x) = \bar{\phi}(\bar{x})$$

4) The partial derivative of a scalar field is a covariant vector:

$$\bar{v}_\rho \equiv \frac{\partial \bar{\phi}}{\partial \bar{x}^\rho} = \frac{\partial \phi}{\partial x^\mu} \frac{\partial x^\mu}{\partial \bar{x}^\rho} = v_\mu \frac{\partial x^\mu}{\partial \bar{x}^\rho}$$

Rules for manipulating tensors:

1) Products of tensors are again tensors

E.g.: let $A_\mu{}^\nu$, $B_{\varrho\sigma}$ be tensors. Then $(A_\mu{}^\nu B_{\varrho\sigma})$ is also a tensor.

2) Tensors of lower rank can be obtained by contracting one upper and one lower index:

if $A_\mu{}^\nu$ is a tensor $A \equiv A_\mu{}^\mu$ is a scalar

3) Lowering or raising an index with the metric or inverse metric again gives a tensor (usually denoted by the same symbol):

$$B^{\mu\nu} \equiv g^{\mu\lambda} B_\lambda{}^\nu$$

note

$$g_{\alpha\mu} B^{\mu\nu} = \underbrace{g_{\alpha\mu}\, g^{\mu\lambda}}_{\delta_\alpha{}^\lambda} B_\lambda{}^\nu = B_\alpha{}^\nu$$

gives back the original mixed tensor.

The affine connection / Christoffel symbol

- goal:

  find the generalization of the e.o.m describing the world line of a freely falling particle in Minkowski space:

$$\frac{d^2 x^\alpha}{d u^2} = 0 \qquad (*)$$

to a spacetime with metric $g_{\alpha\beta}(x)$.

$\Rightarrow$ perform a coordinate transformation on $(*)$:

$$\frac{d^2 \bar{x}^\alpha}{d u^2} = \frac{d}{du}\left(\frac{\partial \bar{x}^s}{\partial x^\mu} \frac{d x^\mu}{d u} \right)$$

$$= \underbrace{\frac{\partial \bar{x}^s}{\partial x^\mu} \frac{d^2 x^\mu}{d u^2}}_{\text{Transformation of a covariant vector}} + \underbrace{\frac{\partial^2 \bar{x}^s}{\partial x^\mu \partial x^\nu} \frac{d x^\nu}{d u} \frac{d x^\mu}{d u}}_{\text{extra piece}} \qquad (**)$$

$\Rightarrow$ $(*)$ is not a tensorial equation. Its form is not invariant under a change of coordinates.

In order to obtain a tensorial equation, we introduce the affine connection (Christoffel symbol) $\Gamma^\lambda{}_{\mu\nu}(x)$:

defining properties:

1) needs to vanish in a coordinate system that is locally

cartesian (recover SR!) ; i.e.

$$\Gamma^{\lambda}{}_{\mu\nu}\Big|_{x_p} = 0 \qquad \text{if} \qquad g_{\mu\nu}(x_p) = \eta_{\mu\nu}$$

$$\partial_\alpha g_{\mu\nu}\Big|_{x_p} = 0$$

2) The transformation law should cancel the extra term in (**):

$$\bar{\Gamma}^{\tau}{}_{\rho\sigma} = \frac{\partial \bar{x}^{\tau}}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial \bar{x}^{\rho}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\sigma}} \Gamma^{\lambda}{}_{\mu\nu} - \frac{\partial^2 \bar{x}^{\tau}}{\partial x^{\mu} \partial x^{\nu}} \frac{\partial x^{\mu}}{\partial \bar{x}^{\sigma}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\rho}} \qquad (3)$$

note: $\Gamma^{\lambda}{}_{\mu\nu}$ is not a tensor!

Combining eqs. 2 and 3 leads to a good tensorial equation:

$$\boxed{\text{Geodesic equation:} \quad \frac{d^2 x^{\lambda}}{du^2} + \Gamma^{\lambda}{}_{\mu\nu} \frac{dx^{\mu}}{du} \frac{dx^{\nu}}{du} = 0}$$

The explicit form of the Christoffel connection can be obtained from the metric $g_{\alpha\beta}(x)$:

$$\Gamma^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\sigma} \left[\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right]$$

Properties:

- $\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu}$ is symmetric in lower indices (by construction)
- $\Gamma^{\lambda}{}_{\mu\nu}\Big|_{x_p} = 0 \quad$ if $\quad \dfrac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\Big|_{x_p} = 0$

Thus the geodesic equation reduces to the e.o.m of S.R if the coordinates are locally cartesian.

Covariant derivatives

- purpose: A covariant derivative allows to construct derivatives of tensors which again obey a tensorial transformation law.

Illustrative example:

- consider a contravariant vector field v^μ;

  under coordinate transformations $\bar{x}^\mu(x^\alpha)$:

$$\bar{v}^{\,s} = \frac{\partial \bar{x}^{s}}{\partial x^\mu} v^\mu$$

Take a partial derivative w.r.t $\bar{x}^\sigma$:

$$\frac{\partial \bar{v}^{\,s}}{\partial \bar{x}^\sigma} = \underbrace{\frac{\partial \bar{x}^{s}}{\partial x^\mu} \frac{\partial v^\mu}{\partial x^\lambda} \frac{\partial x^\lambda}{\partial \bar{x}^\sigma}}_{\text{Tensorial transformation}} + \underbrace{\frac{\partial^2 \bar{x}^{\sigma}}{\partial x^\mu \partial x^\lambda} \frac{\partial x^\lambda}{\partial \bar{x}^\sigma} v^\mu}_{\text{non-tensorial extra piece}}$$

observations:

- The partial derivative of a covariant derivative does not transform as a tensor
- The "extra piece" has the same form as the non-

tensorial part appearing in the transformation of the affine connection !

$\Rightarrow$ adding a suitable connection term we can construct an object transforming as a tensor :

Def. : covariant derivative

The covariant derivative of a vector is defined as

$$D_\nu v^\mu \equiv \partial_\nu v^\mu + \Gamma^\mu{}_{\nu\lambda} v^\lambda$$

Under a change of coordinates it transforms as

$$\bar{D}_\nu \bar{v}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial \bar{x}^\nu} D_\sigma v^\rho$$

General properties :

- The covariant derivative acting on mixed tensors is defined via :

$$D_\alpha A^\mu{}_\nu = \frac{\partial A^\mu{}_\nu}{\partial x^\alpha} + \underbrace{\Gamma^\mu{}_{\alpha\lambda} A^\lambda{}_\nu}_{\text{Part 1}} \underbrace{- \Gamma^\lambda{}_{\nu\alpha} A^\mu{}_\lambda}_{\text{Part 2}}$$

each upper index picks up a term like Part 1 with a " + " sign

each lower index picks up a term like Part 2 with a " − " sign

The covariant derivative has all properties of a derivative:

- product rule (A, B are tensors of arbitrary rank)

$$D_\mu (A \cdot B) = (D_\mu A) \cdot B + A (D_\mu B)$$

- note: covariant derivatives do not commute $D_\mu D_\nu v^\alpha \neq D_\nu D_\mu v^\alpha$

The comma - semicolon notation

- a standard convention abbreviates a partial derivative with a comma:

$$\frac{\partial v^\mu}{\partial x^\nu} \equiv v^\mu{}_{,\nu}$$

- covariant derivatives acting on a tensor are denoted by a semicolon

$$D_\nu v^\mu \equiv v^\mu{}_{;\nu}$$

Covariant volume integrals

- standard problem: integrate a mass density $\rho(x)$ over a volume V

in $\mathbb{R}^3$:

$$m = \int_V d^3x \; \rho(x)$$

Observation: the infinitesimal volume element d^dx $(d = 3, 4 \ldots)$ does not transform as a scalar under coordinate transformations

$$d^d\bar{x} = \left| \frac{\partial \bar{x}}{\partial x} \right| d^dx$$

- d: dimension of integration volume
- $\left| \frac{\partial \bar{x}}{\partial x} \right|$: absolute value of the determinant of the Jacobian $\frac{\partial \bar{x}^\mu}{\partial x^\nu}$ $(\Leftarrow d \times d$ matrix!)

goal: add factor to volume elements to compensate the Jacobian.

guideline

- transformation of metric tensor:

$$\bar{g}_{\rho\sigma} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \bar{x}^\rho} \frac{\partial x^\nu}{\partial \bar{x}^\sigma}$$

- determinant of expression:

$$|\det \bar{g}| = |\det g| \left| \frac{\partial x}{\partial \bar{x}} \right|^2$$

$\Rightarrow$ Jacobian can be compensated by adding a suitable power of the determinant of the metric:

- The combination $d^d x \sqrt{|\det g|}$ transforms as a scalar under coordinate transformations:

$$d^d \bar{x} \sqrt{|\det \bar{g}|} = d^d x \sqrt{|\det g|} \qquad (4)$$

remarks:

- including the absolute value of the determinant allows to write a uniform formula applicable to Lorentzian signature $\det g < 0$ and Euclidean signature $\det g > 0$.

- (4) is a good integration measure when introducing the action principle in general relativity. The Lagrangian density then has to transform as a scalar quantity to realise invariance under general coordinate transformations.

Example: Integration measures on $\mathbb{R}^2$:

Cartesian coordinates

$$ds^2 = dx^2 + dy^2$$

$$g_{\alpha\beta} = \mathrm{diag}[1, 1]$$

$$|\det g| = 1$$

$$\int d^2x \, |\det g|^{1/2} = \int dx \, dy$$

cylindrical coordinates

$$ds^2 = dr^2 + r^2 d\varphi^2$$

$$g_{\alpha\beta} = \mathrm{diag}[1, r^2]$$

$$|\det g| = r^2$$

$$\int d^2x \, |\det g|^{1/2} = \int r \, dr \, d\varphi$$

Thus the universal integration formula recovers known results.

Concluding remarks:

- In this chapter we solved one of Einstein's central problems when formulating general relativity: for him it took years to master the tensor calculus which we assembled in our crashcourse on tensors.

Example : Geodesic motion in $\mathbb{R}^2$

This example serves the purpose of illustrating the concept of geodesic motion and Christoffel symbols.

- Metrics on $\mathbb{R}^2$:

cartesian coordinates : $ds^2 = dx^2 + dy^2$

$$g_{\mu\nu} = \mathrm{diag}(1, 1)$$

cylindrical coordinates : $ds^2 = dr^2 + r^2 d\varphi^2$

$$\bar{g}_{\mu\nu} = \mathrm{diag}(1, r^2)$$

$$\bar{g}^{\mu\nu} = \mathrm{diag}(1, r^{-2})$$

Compute the Christoffel symbols:

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu})$$

Cartesian coordinates:

$$g_{\mu\nu,\alpha} = 0 \qquad \forall \mu, \nu, \alpha$$

$$\Rightarrow \quad \Gamma^{\alpha}{}_{\mu\nu} = 0$$

- cylindrical coordinates:

case $\mu = r$:

$$\Gamma^{r}{}_{\alpha\beta} = \frac{1}{2} \underbrace{g^{rr}}_{=1} (\underbrace{g_{\alpha r,\beta}}_{=0} + \underbrace{g_{\beta r,\alpha}}_{=0} - \underbrace{g_{\alpha\beta,r}}_{\neq 0 \text{ if } \alpha,\beta = \varphi})$$

Thus

$$\Gamma^{r}{}_{\varphi\varphi} = -r$$

case $\mu = \varphi$:

$$\Gamma^{\varphi}{}_{\alpha\beta} = \frac{1}{2} \underbrace{g^{\varphi\varphi}}_{r^{-2}} \left(\underbrace{g_{\alpha\varphi,\beta}}_{\substack{\neq 0 \text{ if} \\ \alpha = \varphi,\ \beta = r}} + g_{\beta\varphi,\alpha} - \underbrace{g_{\alpha\beta,\varphi}}_{=0} \right)$$

Thus :

$$\Gamma^{\varphi}{}_{\varphi r} = \frac{1}{r}$$

observe :

- Christoffel symbols do not transform as tensors: In cartesian coordinates they vanish while in cylindrical coordinates they are non-zero.

Geodesic of a freely falling particle along the x-axis:

- Initial conditions :

$$x^{\dot\alpha}(\tau = 0) = (0, 0) \qquad \left.\frac{dx^{\dot\alpha}}{d\tau}\right|_{\tau=0} = (1, 0)$$

- EOM in Cartesian coordinates :

$$\frac{d^2 x^{\dot\alpha}}{d\tau^2} + \underbrace{\Gamma^{\dot\alpha}{}_{\beta\gamma}}_{=0} \frac{dx^{\beta}}{d\tau} \frac{dx^{\gamma}}{d\tau} = 0$$

solution

$$x^{\alpha}(\tau) = (\tau, 0)$$

Cylindrical coordinates :

- initial conditions :

$$r\,(\bar{J} = 0) = 0 \quad , \qquad \varphi(\bar{J} = 0) = 0$$

$$\left.\frac{dr}{d\bar{J}}\right|_{\bar{J}=0} = 1 \qquad \underbrace{\left.\frac{d\varphi}{d\bar{J}}\right|_{\bar{J}=0} = 0}_{\text{motion is radial}}$$

- EOM in cylindrical coordinates :

$$\frac{d^2 r}{d\bar{J}} = r\,\frac{d\varphi}{d\bar{J}}\,\frac{d\varphi}{d\bar{J}}$$

$$\frac{d^2\varphi}{d\bar{J}} = -\frac{2}{r}\,\frac{d\varphi}{d\bar{J}}\,\frac{dr}{d\bar{J}}$$

- observe if $\frac{d\varphi}{d\bar{J}} = 0$ initially it remains zero for all times !

Thus E.O.M. reduce to

$$\frac{d^2 r}{d\bar{J}} = 0$$

solution

$$r(\bar{J}) = \bar{J} \quad , \qquad \varphi(\bar{J}) = 0$$

describes the same trajectory of the particle in a different coordinate system !