3. Einstein's equations

Summary :

- general relativity encodes the gravitational force in the metric field $g_{\mu\nu}(x)$
- the motion of a freely falling particle in a spacetime with metric $g_{\mu\nu}(x)$ is given by solutions of the geodesic equation

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

with Christoffel symbols:

$$\Gamma^\alpha{}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta} \left(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta} \right)$$

Missing :

- equation describing the dynamics of the metric field
- the corresponding field equations are Einsteins equations schematically :

$$\underbrace{\text{" curvature of spacetime "}}_{\text{gravitational field}} = \underbrace{\text{" energy content "}}_{\text{star}}$$

- goal : make this heuristic idea mathematically precise

Strategy :

- we do not follow the historical path of Einstein
- we follow the construction by Hilbert :
  - remark :

    Hilbert was one of the leading mathematicians working on differential geometry, and there is the rumor that he derived Einsteins equations before Einstein but hesitated to publish his results.

Building blocks :

1) action principle :

   Solutions of the classical equations of motion are extrema of the underlying action

2) The field equations should retain their form under a change of coordinate system, i.e., we are looking for an equation of motion formulated in terms of tensors.

3) The construction should reduce to Newton's gravitational laws in the non-relativistic weak-field limit.

Information from the weak - field limit :

- The metric field outside a spherically symmetric star of mass M is given by the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

notice :

The line element contains the Newtonian gravitational potential

$$\phi = -\frac{GM}{r}$$

$$ds^2 = -(1 + 2\phi) dt^2 + (1 + 2\phi)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

If gravity is weak, $\phi << 1$, and we can expand in ϕ to obtain the static weak - field metric:

$$ds^2 \simeq -(1 + 2\phi) dt^2 + (1 - 2\phi) dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

- note that, contrary to the Schwarzschild metric the static weak - field metric is not an exact solution of Einstein's equations.

Newtonian theory :

- Newtonian gravitational potential ϕ created by a mass - distribution $\rho(\vec{x})$ follows from the Poisson equation :

$$\delta^{ij}\, \partial_i \partial_j \underbrace{\phi}_{\substack{\text{spacetime}\\ \text{metric}}} = 4\pi G\, \underbrace{\rho(\vec{x})}_{\substack{\text{energy}\\ \text{content}}}$$

Takeaway idea:

- The newtonian analysis suggests Einsteins equations should contain two derivatives of the metric field.

An action functional for gravity:

Ansatz:

$$S = \underbrace{\int d^4x \, \sqrt{-g}}_{\text{physical volume element}} \cdot \mathcal{L}$$

Lagrangian density $\mathcal{L}$:

- transforms as scalar under coordinate transformations
- contains up to two derivatives of the metric

Hilbert's conclusion:

$\Rightarrow$ $\mathcal{L}$ should be constructed from curvature tensors!

List of candidates:

1) Riemann tensor:

$$R^{\mu}{}_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^{\mu}{}_{\nu\beta} - \partial_\beta \Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\mu}{}_{\lambda\alpha} \Gamma^{\lambda}{}_{\nu\beta} - \Gamma^{\mu}{}_{\lambda\beta} \Gamma^{\lambda}{}_{\nu\alpha}$$

Properties of the Riemann tensor:

1) $R^{\mu}{}_{\nu\alpha\beta}$ is antisymmetric in the last two indices

$$R^{\mu}{}_{\nu\alpha\beta} = - R^{\mu}{}_{\nu\beta\alpha}$$

- read off from the definition.

2) Satisfies the first Bianchi identity:

$$R^{\mu}{}_{\nu\alpha\beta} + R^{\mu}{}_{\alpha\beta\nu} + R^{\mu}{}_{\beta\nu\alpha} = 0$$

3) Satisfies the second Bianchi identity:

$$D_{\lambda} R^{\mu}{}_{\nu\alpha\beta} + D_{\alpha} R^{\mu}{}_{\nu\beta\lambda} + D_{\beta} R^{\mu}{}_{\nu\lambda\alpha} = 0$$

4) $R_{\mu\nu\alpha\beta} \equiv g_{\mu\lambda} R^{\lambda}{}_{\nu\alpha\beta}$

is symmetric under the exchange of the first and second index pair

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$$

Caveat:

- $R^{\mu}{}_{\nu\alpha\beta}$ does not transform as a scalar under coordinate transformations

$\Rightarrow$ contract suitable index pairs to obtain tensors of lower rank!

2) <u>Ricci tensor $R_{\mu\nu}$:</u>

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}$$

<u>Properties</u>

1) symmetric: $R_{\mu\nu} = R_{\nu\mu}$

- follows from property 4) of the Riemann tensor

2) computable from the Christoffel symbol:

$$R_{\mu\nu} = \partial_\lambda \Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}{}_{\lambda\mu} + \Gamma^{\lambda}{}_{\sigma\lambda} \Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\sigma\nu} \Gamma^{\sigma}{}_{\lambda\mu}$$

Caveat:

- The Ricci tensor is also not a scalar

3) <u>Ricci scalar R:</u>

$$R \equiv g^{\mu\nu} R_{\mu\nu}$$

- scalar quantity $\Rightarrow$ good candidate for $\mathcal{L}$

$\Rightarrow$ choice is unique!

Thus we arrive at the Einstein - Hilbert action:

$$S^{EH} \equiv \frac{1}{16\pi G} \int d^4x \, \sqrt{-g} \, (R - 2\Lambda)$$

remarks :

- we added the "zero derivative" term Λ which corresponds to a cosmological constant.
- the prefactor $\frac{1}{16\pi G}$ with G being Newton's constant gives the correct coupling between matter and curvature (compare to the Poisson equation).
- matter can easily be included by adding terms to the action :

$$S^{\text{total}} = S^{EH} + S^{\text{matter}}$$

example :

- real scalar field minimally coupled to gravity :

$$S^{\text{scalar}} = -\int d^4x \sqrt{-g} \Big(\underbrace{\tfrac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi}_{\text{kinetic term}} + \underbrace{V(\phi)}_{\text{potential}} \Big)$$

- since S^{EH} is invariant under a change of coordinates the variational principle leads to a tensorial equation.

Einstein's equations from the variational principle

- aim : compute

$$\frac{1}{\sqrt{-g}} \frac{\delta S^{EH}}{\delta g^{\mu\nu}} \overset{!}{=} 0$$

setting the first variation to zero implies we obtain an extremum of the action.

Interlude : variations with respect to fields:

- prototype : partial derivatives :

$$\frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu$$

- analogously, the variation with respect to a field is defined as:

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \delta^4(x-y)$$

where $\delta^4(x-y)$ is the 4-dimensional delta-distribution satisfying

$$\int d^4x \; f(x) \; \delta^4(x-y) = f(y)$$

If the field carries additional indices, these also appear in the variation and have to agree :

$$\frac{\delta g_{\mu\nu}(x)}{\delta g_{\alpha\beta}(y)} = \frac{1}{2} \left(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\beta_\mu \delta^\alpha_\nu \right) \delta^4(x-y)$$

Variation of S^{EH} with respect to $g_{\mu\nu}$ uses the product rule:

$$\delta S^{EH} = \frac{1}{16\pi G}\int d^4x \Big[(\delta\sqrt{-g})\,(R - 2\Lambda) + \sqrt{-g}\,(\delta g^{\mu\nu})\,R_{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\,\delta R_{\mu\nu}\Big]$$

need the variation of the building blocks:

- denote the variation of the metric by $\delta g_{\mu\nu}$

1) <u>Variation of the inverse metric:</u>

$$\delta g^{\mu\nu} = - g^{\mu\alpha}\, g^{\nu\beta}\, \delta g_{\alpha\beta}$$

Proof:

$$\begin{aligned} 0 &= \delta(\delta_\mu{}^\nu) \\ &= \delta(g_{\mu\lambda}\, g^{\lambda\nu}) \\ &= (\delta g_{\mu\lambda})\, g^{\lambda\nu} + g_{\mu\lambda}\,(\delta g^{\lambda\nu}) \end{aligned}$$

contract with $g^{\sigma\mu}$ yields

$$\begin{aligned} g^{\sigma\mu}\, g^{\lambda\nu}\, \delta g_{\mu\lambda} &= - \underbrace{g^{\sigma\mu}\, g_{\mu\lambda}}_{= \delta^\sigma{}_\lambda}\, \delta g^{\lambda\nu} \\ &= - \delta g^{\sigma\nu} \end{aligned}$$

□

2) <u>Variation of a determinant $g = \det g_{\mu\nu}$:</u>

$$\delta g = g \, g^{\mu\nu} \, \delta g_{\mu\nu}$$

and from the chain rule

$$\delta \sqrt{-g} = -\frac{1}{2} \frac{1}{\sqrt{-g}} \delta g$$

$$= \frac{1}{2} \sqrt{-g} \, g^{\mu\nu} \, \delta g_{\mu\nu}$$

Proof of the first identity uses that for a matrix M:

$$\ln \det M = \operatorname{Tr} \ln M$$

where $\ln M$ is defined as $\exp(\ln M) = M$

apply variation:

$$\frac{1}{\det M} \delta \det M = \operatorname{Tr} M^{-1} \delta M$$

rewrite in terms of the metric $g_{\mu\nu}$:

$$\delta g = g \, \underline{g^{\mu\nu} \, \delta g_{\mu\nu}}$$

the trace corresponds to the contraction of indices.

3) <u>Variation of $R_{\mu\nu}$:</u>

$$\delta R_{\mu\nu} = D_\lambda \delta \Gamma^\lambda{}_{\mu\nu} - D_\nu \delta \Gamma^\lambda{}_{\lambda\mu}$$

remarks:

- The proof can be found in Carrols textbook. Since it is technical, we do not repeat it here.

- the variation of $R_{\mu\nu}$ produces surface terms only and does not contribute to the equations of motion.

Substitute building blocks:

$$\delta S^{EH} = \frac{1}{16\pi G} \int d^4x \Big[\left(\frac{1}{2} \sqrt{-g}\, g^{\mu\nu} \delta g_{\mu\nu} \right) (R - 2\Lambda)$$

$$+ \sqrt{-g} \left(- g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu} \right) R_{\alpha\beta}$$

$$+ \text{surface terms} \Big]$$

$$= \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda) \right] \delta g^{\mu\nu}$$

- Here we used $\delta g_{\alpha\beta} = - g_{\alpha\mu} g_{\beta\nu} \delta g^{\mu\nu}$ in the second step.

The gravitational part of Einsteins equations is recovered as

$$\frac{1}{\sqrt{-g(y)}} \frac{\delta S^{EH}}{\delta g^{\alpha\beta}(y)} \overset{!}{=} 0$$

$$= \frac{1}{\sqrt{-g(y)}} \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda) \right] \underbrace{\frac{\delta g^{\mu\nu}}{\delta g^{\alpha\beta}}}_{= \frac{1}{2} \left(\delta^\mu_\alpha \delta^\nu_\beta + \delta^\nu_\beta \delta^\mu_\alpha \right) \delta^4(x-y)}$$

$$= \frac{1}{16\pi G} \frac{1}{\sqrt{-g(y)}} \cdot \sqrt{-g(y)} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda) \right]$$

Thus we recover Einstein's equations in the absence of matter :

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - g_{\mu\nu} \Lambda$$

Including matter :

The stress-energy tensor $T_{\mu\nu}$ is defined from the variation principle via :

$$T_{\mu\nu} \equiv - \frac{2}{\sqrt{-g}} \frac{\delta S^{matter}}{\delta g^{\mu\nu}}$$

applying the variation principle to S^{total} :

$$\frac{1}{\sqrt{-g}} \frac{\delta S^{EH}}{\delta g^{\mu\nu}} + \frac{1}{\sqrt{-g}} \frac{\delta S^{matter}}{\delta g^{\mu\nu}} = 0$$

$$\frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda \right) - \frac{1}{2} T_{\mu\nu} = 0$$

This gives the complete Einstein's equations determining the metric field $g_{\mu\nu}$:

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - g_{\mu\nu} \Lambda + 8\pi G \, T_{\mu\nu}}$$

remarks :

1) The combination

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$

is called the Einstein tensor. From the second Bianchi

$$D_\mu G^{\mu\nu} = 0$$

Consequence: Einstein's equations imply that the stress-energy tensor is conserved:

$$D_\mu T^{\mu\nu} = 0$$

2) Typical examples for stress-energy tensors encountered in cosmology are

- perfect fluids:

  characterized by energy density ρ and pressure p:

  $$T^{\alpha\beta} = (\rho + p)\, u^\alpha u^\beta + p\, g^{\alpha\beta}$$

  where u^α is the 4-velocity of the fluid.

- for a perfect fluid at rest w.r.t. the cosmic coordinates of the FRW metric, this simplifies to

  $$T_\mu{}^\nu = \mathrm{diag}(-\rho, p, p, p)$$

3) The cosmological constant is a special form of a perfect fluid with $\rho = -p$

$$T^{\alpha\beta} = \frac{\Lambda}{8\pi G}\, g^{\alpha\beta}$$

4) Notably Einstein's equations are non-linear (they include the metric $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$)

- much harder to solve than Maxwells equations
- general solution is not known

  ⇒ numerical relativity is a very active research field.

This completes the survey of the theoretical background in general relativity underlying the cosmological model building and our current understanding of the evolution of our universe.