4. The homogeneous and isotropic universe

· Last time (see introductory part):

Einstein's equations provide a field theory for gravity

· dynamical object: spacetime metric $g_{\mu\nu}(x)$

· field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \, T_{\mu\nu}$$

left-hand-side:

· curvature of spacetime:

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \partial_\nu \Gamma^\lambda{}_{\lambda\mu} + \Gamma^\lambda{}_{\sigma\lambda} \Gamma^\sigma{}_{\mu\nu} - \Gamma^\lambda{}_{\sigma\nu} \Gamma^\sigma{}_{\lambda\mu}$$

$$R \equiv g^{\mu\nu} R_{\mu\nu}$$

right-hand-side:

· matter content of the universe encoded in stress-energy tensor $T_{\mu\nu}$

Goal of this lecture:

· use Einstein's equations to construct a simple but very successful cosmological model:
Friedmann-Robertson-Walker (FRW) cosmology

Inventory of the universe

Astrophysical surveys indicate that on largest scales

a) the universe is populated by:

- stars and gas gravitationally bound in galaxies
- diffuse radiation (e.g. the cosmic microwave background)

  remark:

  cosmologists call all relativistic particles "radiation"

- dark matter
  - massive objects of unknown character
- vacuum energy

b) the universe is expanding:

- using type Ia supernovae observations

  (≡ standard candles with equal luminosity)

  measuring their brightness and redshift indicates that distant galaxies move away from us, independent of their direction.

c) averaged over large volumes of O(100 MPc) the universe is

- isotropic (looks the same in any direction)
- homogeneous (there is no prefered point)

experimental evidence:

- observation of a uniform CMB
- large scale galaxy surveys

FRW - cosmology:

- builds a cosmological model on the "cosmological principles" of homogenity and isotropy

remark:

- more complex cosmological models include anisotropies as small perturbations of the FRW - background

Implementation of the "cosmic principles"

$\Rightarrow$ homogenity and isotropy place severe restrictions on the form of spacetime metric $g_{\mu\nu}(x)$

$\Rightarrow$ ansatz for the line element:

- isotropy implies:

  a) the metric cannot contain g_{ti} - terms:

  a term $N_i \, dt \, dx^i$ in the line element would introduce a preferred direction, the vector N_i

  b) the spatial part g_{ij} must be spherically symmetric

- homogeneity implies:

  a) g_{tt} must be independent of the spatial coordinates

  b) g_{ij} must be homogeneous, i.e. the spatial slices do not contain any preferred point.

Result: there are only 3 classes of models respecting homogeneity and isotropy:

FRW line-element:

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \begin{matrix} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{matrix} \quad (d\theta^2 + \sin^2\theta \, d\phi^2)\right]$$

The spatial slices are:

1) closed spheres ($\sin^2\chi$)

2) flat space (χ^2)

3) open hyperboloids ($\sinh^2\chi$)

The scale-factor $a(t)$ depends on the cosmic time t only and determines the physical distance between two points on the spatial slice.

Coordinate transformation unifying the 3 line elements:

closed : $r^2 = \sin^2 \chi$ $\quad k = +1$

flat : $r^2 = \chi^2$ $\quad k = 0$

open : $r^2 = \sinh^2 \chi$ $\quad k = -1$

yield the final form of the FRW line-element:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Admissible stress-energy tensors:

- matter must be modeled as a perfect fluid:
  (any heat-transfer would violate homogeneity, ...)
- energy density ρ and pressure p must be independent of the spatial coordinates (homogeneity):

$$\rho = \rho(t), \qquad p = p(t)$$

- the perfect fluid is at rest w.r.t. the cosmic coordinates
  (any spatial velocity would define a preferred direction)

$\Rightarrow$ four-velocity of fluid : $u^\alpha = (1, 0, 0, 0)$

consequence:
the position of a galaxy is given by the same spatial coordinates at all times.

Evaluating the general stress-energy tensor for these conditions:

$$T^{\mu}{}_{\nu} = \text{diag}\left[-\rho(t),\ p(t),\ p(t),\ p(t)\right]$$

Causal structure of the FRW-universe

- How do particles and light propagate in the FRW spacetime?

$\Rightarrow$ construct the light cones of the geometry ($ds^2 = 0$)

Perform a change of time coordinate

- cosmic time $t \Rightarrow$ conformal time η

Definition of η:

$$\eta = \int \frac{dt}{a(t)} \quad , \quad d\eta = \frac{1}{a}\, dt$$

Line element in conformal time:

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\chi^2 + \phi_k(\chi)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

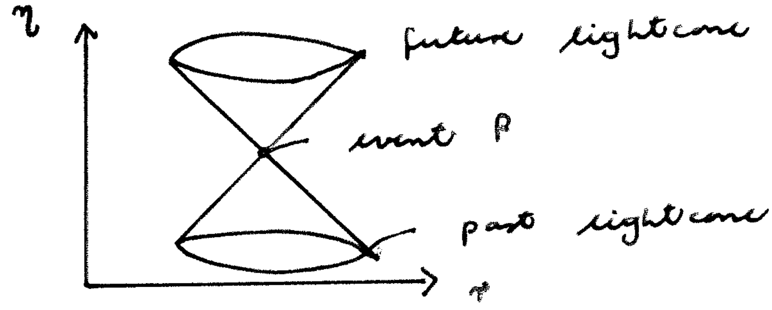
isotropic universe:

$\Rightarrow$ restrict to light rays propagating radially:

$$ds^2 = a(\eta)^2\left[-d\eta^2 + d\chi^2\right]$$

$\Rightarrow$ in conformal time the light cone structure is identical

to the one of Minkowski space:



Observation:

- not all points of the FRW spacetime are in causal contact $\Rightarrow$ spacetime has horizons!

<u>Def: particle horizon:</u>

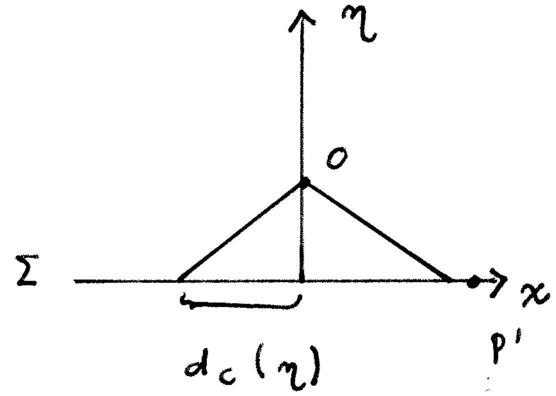
- the maximum comoving (coordinate) distance light can travel between initial time t_i and time t is determined by the conformal time interval:

$$d_c(\eta) = \eta - \eta_i = \int_{t_i}^{t} \frac{dt}{a(t)}$$

This corresponds to the physical distance

$$d_p(t) = a(t)\, d_c(t)$$

note: $2\, d_c(\eta)$ is the coordinate interval where information about the cosmic fluid can be seen at time η:



Information at point P' cannot reach the observer O

$\Rightarrow$ particle horizon

limits the events that can be seen by an observer

- Puzzle :
  - as η increases, more and more of the hypersurface Σ becomes visible. This includes patches that have not been in causal contact in the past. How come we observe an homogeneous universe ?

Dynamics of the model :

- obtained by evaluating Einstein 's equation

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \, T_{\mu\nu} \qquad (1)$$

and energy - momentum conservation

$$D_\mu T^{\mu\nu} = 0 \qquad (2)$$

- for the FRW - ansatz :

- Result (derived using mathematica notebook !)

Friedman - equations :

$$\boxed{\begin{array}{lll} G_{00} : & \left(\frac{\dot a}{a}\right)^2 - \frac{8\pi G \rho}{3} = - \frac{k}{a^2} \\[2ex] G_{ii} : & \frac{\ddot a}{a} = - \frac{4}{3}\pi G \left(\rho + 3p \right) \\[2ex] D_\mu T^{\mu\nu} : & \frac{d}{dt} \left(\rho a^3 \right) = - p \frac{d}{dt} a^3 \end{array}}$$

Consequence of the Bianchi - identity :

- equations are not independent

(2) is implied by (1) and (3) :

From energy-momentum conservation: $(G = 1)$

$$\dot{\rho}\, a^3 + 3 \rho\, a^2 \dot{a} = - p\, 3 a^2 \dot{a}$$

$$\dot{\rho} = -3 \frac{\dot{a}}{a} (\rho + p) \qquad (*)$$

Take time-derivative of first equation:

$$\dot{a}^2 - \frac{8\pi\rho}{3} a^2 = -k$$

$$2 \dot{a} \ddot{a} - \frac{8\pi}{3} \rho\, 2 a \dot{a} - \frac{8\pi}{3} a^2 \dot{\rho} = 0$$

$$2 \dot{a} \left(\ddot{a} + \frac{4\pi}{3} a (\rho + 3p) \right) = 0$$

where we eliminated $\dot{\rho}$ using $(*)$

The bracket is the second equation.

It is useful to rewrite the continuity equation in terms of the Hubble parameter:

$$\dot{\rho} + 3H(\rho + p) = 0$$

Solving the FRW equations

- assume : matter content satisfies an equation of state relating pressure and energy densities

$$p = w \rho \qquad (*)$$

Examples :

$w = 0$: pressureless dust (galaxies, non-relativ. particles)

$w = 1/3$: radiation (photons, relativistic particles)

$w = -1$: cosmological constant

cover the most important cases

Step 1 : solve the continuity equation :

using $(*)$:

$$\dot{\rho} + 3 \rho \frac{\dot{a}}{a} (1 + w) = 0$$

or

$$\frac{d \ln \rho}{d \ln a} = -3 (1 + w)$$

has solutions :

$w \neq -1$: $\rho \propto a^{-3(1+w)}$

$w = -1$: $\rho = \text{const.}$

Fixing the initial conditions (the canonical choice):

- denote t_0 as time today

and $\quad \rho_0 \equiv \rho(t_0) \quad$ energy density today

$\quad a_0 \equiv 1 \quad$ scale factor today

Then the continuity equation is solved by:

$$\rho(t) = \rho_0 \, a(t)^{-3(1+w)}$$

□

<u>Step 2: determine the dynamics of the scale factor:</u>

- Friedmann equation (with $k = 0$ for simplicity)

$$\left(\frac{\dot{a}}{a} \right)^2 - \frac{8\pi G}{3} \rho(a) = 0$$

substitute $\rho(a)$:

$$\left(\frac{\dot{a}}{a} \right)^2 - \frac{8\pi G}{3} \rho_0 \, a^{-3(1+w)} = 0$$

case: $w = -1$: (cosmological constant only)

$$H = \frac{\dot{a}}{a} = \text{constant}$$

Solution

$$a(t) = a_0 \, e^{Ht}$$

- This is the famous de Sitter solution of GR.

case $w \neq -1$:

$$\frac{da}{dt} = \sqrt{\frac{8\pi G}{3}\rho_0}\; a^{-\frac{1}{2}(1+3w)}$$

This is a separable ordinary differential equation with solution

$$\int da \; a^{\frac{1}{2}(1+3w)} = \sqrt{\frac{8\pi G}{3}\rho_0}\int dt$$

Thus

$$a(t)^{\frac{3}{2}(1+w)} \propto t$$

Fixing initial condition $a_0 = a(t_0) = 1$:

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}$$

Thus we have determined the complete evolution of a FRW universe whose matter content is dominated by one particular type of "matter" specified by the equation of state

Summary : (most important contributions to $T^{\mu\nu}$ in cosmology) :

	w	$\rho(a)$	$a(t)$	$a(\eta)$	η_i
matter dominated (MD)	0	a^{-3}	$t^{2/3}$	η^2	0
radiation dominated (RD)	$1/3$	a^{-4}	$t^{1/2}$	η	0
cosmological const.	-1	a^0	e^{Ht}	$-\eta^{-1}$	$-\infty$

The critical energy density ρ_{crit} :

- evaluate flat Friedmann equation today :

$$H_0^2 - \frac{8\pi G}{3} \rho_0 = 0$$

Implies that the universe must be at critical density :

$$\rho_{crit} = \frac{3}{8\pi G} H_0^2$$

measurement :

$$\rho_{crit} = 1.88 \times 10^{-29} \; h^2 \; g/cm^3$$

(corresponds to approx 1 proton per liter of universe)

The cosmological standard model (ΛCDM)

- expect that the realistic universe is populated by more than one species (baryons, photons, dark matter, ...) all contributing to ρ_{total} with their own equation of state :

$$\rho_{total} = \sum_i \rho^i \qquad p_{total} = \sum_i p^i \qquad w_i \equiv \frac{p^i}{\rho^i}$$

useful to define their energy densities relative to the critical density :

$$\Omega_i = \frac{\rho_0^i}{\rho_{crit}}$$

For $k \neq 0$ there is also an energy density associated with the spatial curvature :

$$\Omega_k = - \frac{k}{a_0^2 H_0^2}$$

With these definitions, the Friedmann equation can be cast into the form

$$\left(\frac{H}{H_0} \right)^2 = \sum_i \Omega_i \, a^{-3(1+w_i)} + \frac{\Omega_k}{a^2} \qquad (**)$$

Evaluating at $t = t_0$ gives a constraint on the sum of the relative energy densities:

$$\sum_i \Omega_i + \Omega_k = 1$$

<u>Matter content of our universe</u>:

- combining cosmological observations (detailed experiments will be described by other lecturers) yields:

$$\Omega_b = 0.04 \quad , \quad \Omega_{DM} = 0.23 \qquad \Omega_\Lambda = 0.72$$

- visible baryonic matter makes only 4% of the energy budget in our universe !
- no evidence for spatial curvature $\Omega_k \approx 0$!

scientific challenge :

- understand Ω_{DM} !
  - matter not contained in the standard model of particle physics
  - interacts only weakly with visible matter
    (in particular it has no electric charge coupling to photons, hence the name "dark")
  - structure formation : it must be non-relativistic

$\Rightarrow$ This leaves a plethora of theoretical options

$\Rightarrow$ Direct searches are under way but have not yielded conclusive results yet.

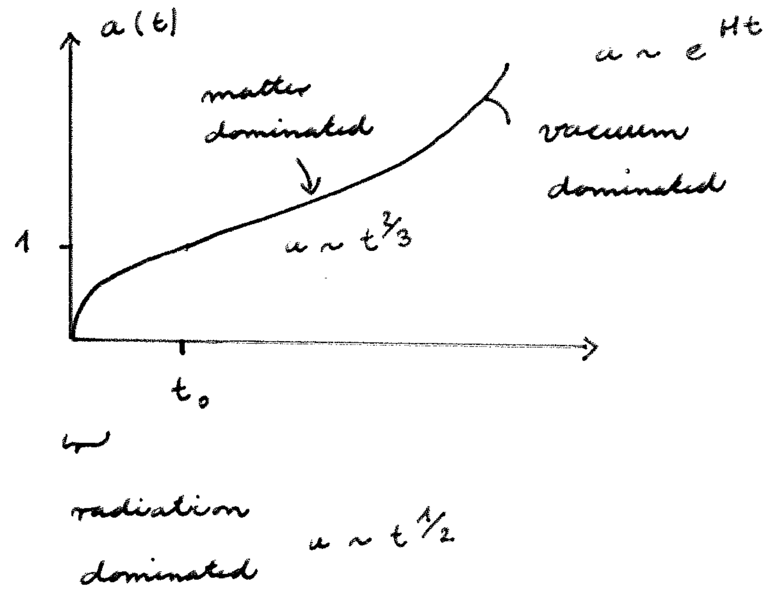
Evolution of a realistic universe containing:

radiation : $\Omega_r = 1/3$

non-relativistic matter : $\Omega_m = 1/3$

vacuum energy (cosmological constant) : $\Omega_\Lambda = 1/3$

dilution of ρ_i during the expansion of the universe (see table, eq (**)) has 3 phases :



remarks:

- compared to this example our universe has:
  - less radiation
  - more vacuum energy

- for $t \to 0$ the scale factor becomes zero

  while $S(a) \to \infty$

  in this stage the universe is infinitely dense

  $\Rightarrow$ this is called the Big bang singularity.