6. Density fluctuations and power spectra

- big success of inflation
  - correct prediction of cosmological density fluctuations observed today

  ⇒ this established inflation as a commonly accepted epoch in the evolution of the early universe

Goal of this chapter:

- understand the theory behind this extraordinary success

Summary of the key results:

- the comoving Hubble radius shrinks (quasi exponentially) during inflation
- fluctuations can be organized / labelled by their comoving wave number $k = \frac{2\pi}{L}$, L being the comoving wavelength

  the modes are called
  - superhorizon if $k < aH$ $\quad \left(L > \frac{2\pi}{aH} \right)$
  - subhorizon if $k > aH$ $\quad \left(L < \frac{2\pi}{aH} \right)$

notation:

- we use the subscript $*$ to denote that a quantity is evaluated when the mode with wavenumber k crosses the horizon $k = a_* H_*$

Crucial property:

if a mode exits the horizon it is described by a classical probability distribution with variance given by its power spectrum at the horizon crossing

Two power spectra playing a role in cosmological observations:

- comoving curvature perturbations $\mathcal{R}$

  (equivalently fluctuations in the inflaton field $\delta\phi$)

  satisfy

  $$\langle \mathcal{R}_{\vec{k}} \, \mathcal{R}_{\vec{k}'} \rangle = (2\pi)^3 \, \delta^3(k + k') \, P_{\mathcal{R}}(k)$$

  - $\langle \dots \rangle$ denotes the ensemble average
  - $\delta^3(\vec{k} + \vec{k}')$ indicates that Fourier-modes with different $\vec{k}$ are uncorrelated / decoupled

  ∴ power spectrum $P_{\mathcal{R}}(k) = \dfrac{H^2}{2k^3} \dfrac{H^2}{\dot{\phi}^2} \Big|_{k = aH}$

- tensor perturbations (fluctuations in the spatial metric) with power spectrum

  $$P_t(k) = \frac{4}{k^3} \frac{H^2}{M_{Pl}^2} \Big|_{k = aH}$$

Typically, the power spectra are rescaled as

$$\Delta_i^2(k) \equiv \frac{k^3}{2\pi^2} P_i(k)$$

$i = \{ \mathcal{R}, t \}$

Based on $\Delta_{\mathcal{R}}^2(k) \equiv \Delta_S^2(k)$ one defines the spectral index

$$n_S - 1 \equiv \frac{d \ln \Delta_S^2}{d \ln k}$$

The scale invariant spectrum (Harrison - Zeldovic spectrum) corresponds to $n_S = 1$

In addition, one has the running of the spectral index

$$\alpha_S \equiv \frac{d n_S}{d \ln k}$$

The spectral index for tensor fluctuations n_t is defined by

$$n_t \equiv \frac{d \ln \Delta_t^2}{d \ln k}$$

Furthermore one defines the scalar - to - tensor ratio

$$r \equiv \frac{\Delta_t^2}{\Delta_S^2}$$

For slow-roll inflation the spectral indices can be computed from the scalar potential and slow-roll parameters

$$\Delta_s^2(k) = \frac{1}{24\pi^2} \frac{V(\phi)}{M_{Pl}^4} \frac{1}{\varepsilon_V} \Bigg|_{k=aH}$$

$$\Delta_t^2(k) = \frac{2}{3\pi^2} \frac{V(\phi)}{M_{Pl}^4} \Bigg|_{k=aH}$$

The spectral index is then related to the slow-roll parameters

$$n_s - 1 \equiv 2\eta_V - 6\varepsilon_V$$

$$n_t = -2\varepsilon_V$$

and

$$r = 16\,\varepsilon_V$$

The Lyth bound allows to relate r to the energy scale of inflation

$$\frac{\Delta\phi}{M_{Pl}} \simeq \left(\frac{r}{0.01}\right)^{1/2}$$

remark :

- a large value of r indicates that inflation happened at transplanckian energies !

Comparison with experimental data :

- Current values for n_s combine

  WMAP 9 + Baryon acoustic peaks + high resolution CMB data (eCMB)

$$n_s = 0.958 \pm 0.008$$

$\Rightarrow$ almost flat power - spectrum

$\Rightarrow$ in agreement with slow - roll inflation

remarks :

- measurements for n_t and r are under way
  - hitherto : no detection of primordial grav. waves
    : only upper bound on r

Goal of remaining chapter :

- understand where the results quoted in this summary come from.

6.1. Foundations of cosmic perturbation theory

- The homogeneous and isotropic FRW solutions provide a good approximation of the observed universe

Question :

- can we obtain a better description providing more predictions which can be compared to observations ?

Idea :

- use homogeneous and isotropic solutions as a background $\bar{X}(t)$ (note : $\bar{X}(t)$ contains the background metric $\bar{g}_{\mu\nu}(t)$ and stress - energy tensor $\bar{T}_{\mu\nu}(t)$)

- study the evolution of small perturbations $\delta X(t, \vec{x})$ around background :

$$\delta X(t, \vec{x}) = X(t, \vec{x}) - \bar{X}(t)$$

- note : perturbations depend on spatial coordinates

  $\Rightarrow$ they are not homogeneous and isotropic !

- for our universe (e.g. perturbations in the CMB):

$$\frac{\delta X(t, \vec{x})}{\bar{X}(t)} \lesssim 10^{-5}$$

$\Rightarrow$ expanding in perturbations is a good strategy

Dynamics of perturbations :

- Evaluate Einstein's equations for $X(t, \vec{x})$
- expand result around the background $\bar{X}(t)$ retaining the terms linear in $\delta X(t, \vec{x})$ only :

$\Rightarrow$ Result: linearized Einstein equations provide e.o.m. for the perturbations :

$$\delta G_{\mu\nu} = 8\pi G \; \delta T_{\mu\nu}$$

Important property :

- the linearized Einstein equations are invariant under spatial translations :

$$x^i \to x^i + \Delta x^i \quad , \quad \Delta x^i = \text{const.}$$

consequence of the background being independent of x^i

$\Rightarrow$ use Fourier components to study perturbations

i.e. instead of the dynamics of $\delta X(t, \vec{x})$ formulate the problem in terms of its Fourier - transform

$$X(t, \vec{k}) \equiv \int d^3x \;\; X(t, \vec{x}) \; e^{i \vec{k} \vec{x}}$$

In exercise 1 it is then proven :

- The translation symmetry has the profound consequence that the Fourier - amplitudes $X_{k}(t)$ decouple

$\Rightarrow$ The dynamics of each Fourier mode can be studied independently.

<u>Question: What are the good variables to express the fluctuations?</u>

- complication:
  - physics is independent of the choice of coordinate system
  - the components of $g_{\mu\nu}$ transform non-trivially however
- Strategy:
  - express fluctuations in terms of variables which are invariant under a change of coordinates

    ($\Rightarrow$ gauge invariant variables like the field strength tensor $F_{\mu\nu}$)

Starting point:

From the linearized Einstein equations we conclude:

- we have two types of perturbations
  - metric perturbations $\delta g_{\mu\nu}(t, x)$
  - perturbations of the stress-energy tensor $\delta T_{\mu\nu}(t, x)$

Metric perturbations and the scalar - vector - tensor decomposition

- conventionally perturbations of the flat FRW metric are parameterized by

$$ds^2 = -(1 + 2\Phi)\, dt^2 + 2 a B_i\, dx^i dt$$

$$+ a^2 \left[(1 - 2\Psi)\, \delta_{ij} + E_{ij} \right] dx^i dx^j$$

a^2 : scale factor

Φ : scalar, lapse function

B_i : shift vector

Ψ : scalar, spatial curvature perturbations

E_{ij} : symmetric, traceless 3 - tensor : $E_{ij} = E_{ji}$, $\delta^{ij} E_{ij} = 0$

spatial shear tensor

Further decomposition in helicity states:

- vector: $B_i = \partial_i B - S_i$ $\qquad\qquad \underbrace{\partial^i S_i = 0}_{\text{transverse vector}}$

- tensor:

$$E_{ij} = 2\left(\partial_i \partial_j - \frac{1}{3}\delta_{ij}\partial^2\right) E + \partial_i F_j + \partial_j F_i + h_{ij}$$

where

F_i : $\partial^i F_i = 0$ is a transverse vector

h_{ij} : $\delta^{ij} h_{ij} = 0$, $\partial^i h_{ij} = 0$ transverse - traceless remarks :

- S_i and F_i are not created during inflation. They will not be considered any further
- the transverse traceless part h_{ij} is invariant under coordinate transformations
- the remaining fields can be combined into objects invariant under coordinate transformations

Under the infinitesimal change of coordinates

$$t \to t + \alpha \,, \qquad x^i \to x^i + \delta^{ij} \partial_j \beta$$

The scalar perturbations transform according to

$$\phi \to \phi - \dot{\alpha}$$

$$B \to B + a^{-1} \alpha - a \dot{\beta}$$

$$E \to E - \beta$$

$$\psi \to \psi + H \alpha$$

remark :

- the derivation of these transformation rules uses the invariance of the line element under the coordinate change :

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu$$

Example :

- the transformation behavior of $\tilde{\Phi}$ can be read off from the $(dt)^2$ - term :

$$(1+2\phi)\, dt^2 \rightarrow (1+2\tilde{\phi})\underbrace{(d\tilde{t})^2}$$

$$= (1+2\tilde{\phi})(1+\dot{\alpha})^2 (dt)^2$$

$$= (1+2\tilde{\phi}+2\dot{\alpha})(dt)^2$$

Thus $\tilde{\phi} = \phi - \dot{\alpha}$ yielding the transformation behavior

$$\phi \rightarrow \tilde{\phi} = \phi - \dot{\alpha}$$

Perturbations of the stress - energy tensor

two sources :

- perturbations in the energy density $\delta\rho(t,x)$ and pressure $\delta p(t,x)$
- perturbations in the fluid four - velocity :

$$u_\mu = (-1-\phi\,,\, a v_i)$$

$$u^\mu = (1-\phi\,,\, a^{-1}(v^i - B^i))$$

The perturbed stress - energy tensor is then given by

$$T^0_0 = -(\bar{\rho} + \delta\rho)$$

$$T^0_i = (\bar{\rho} + \bar{p})\, a v_i$$

$$T^i_0 = -(\bar{\rho}+\bar{p})(v^i - B^i)/a$$

$$T^i_j = \delta^i_j(\bar{p} + \delta p) + \Sigma^i_j$$

- anisotropic stress Σ^i_j is traceless $\Sigma^i_i = 0$
- definition of momentum density

$$\partial_i \delta q \equiv (\bar{\rho} + \bar{p}) v_i$$

Transformation under coordinate trafo:

- Σ^i_j is invariant

$$\delta \rho \to \delta \rho - \dot{\bar{\rho}} \alpha$$

$$\delta p \to \delta p - \dot{\bar{p}} \alpha$$

$$\delta q \to \delta q + (\bar{\rho} + \bar{p}) \alpha$$

Quantities invariant under coordinate transformations

- all quantities have a geometrical interpretation

1) Comoving curvature perturbation $\mathcal{R}$:

$$\mathcal{R} = \psi - \frac{H}{\bar{\rho} + \bar{p}} \delta q$$

- geometrically $\mathcal{R}$ is the spatial curvature of slices of constant ϕ
- invariance from transformation laws above

$$\mathcal{R} \to \psi + H\alpha - \frac{H}{\bar{\rho} + \bar{p}} \left(\delta q + (\bar{\rho} + \bar{p}) \alpha \right)$$

$$= \psi - \frac{H}{\bar{\rho} + \bar{p}} \delta q$$

□

- for scalar field inflation:

$$\partial_i \delta q = - \bar{\dot{\phi}} \, \partial_i \delta\phi \quad , \quad \bar{\rho} + \bar{p} = \dot{\bar{\phi}}^2$$

yielding:

$$\mathcal{R} = \Psi + \frac{H}{\dot{\bar{\phi}}} \delta\phi$$

2) non - adiabatic density fluctuations

$$\delta p_{en} \equiv \delta p - \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}} \delta\rho$$

- for single - field inflation models density fluctuations are always adiabatic, i.e. δp_{en}

remark:

- adiabatic implies that locally the perturbed system is identical to the unperturbed system at a different time.

Remarks on gauge - invariant field combinations

- one can define more invariant quantities (measure of inflaton perturbations, curvature perturbations on constant density slices, Bardeen potentials, ...)

  we limit our focus on $\mathcal{R}$.

- picking specific coordinate systems allows to implement "gauge choices". E.g. comoving gauge sets $\delta q = E = 0$ so that $\mathcal{R} = \Psi$

Evolution equation for $\mathcal{R}$:

- analyzing the linearized Einstein equations yields

$$\dot{\mathcal{R}} = - \frac{H}{\bar{\rho} + \bar{p}} \, \delta p_{in} + \frac{k^2}{(aH)^2} \, (\dots)$$

- for adiabatic perturbations $\delta p_{in} = 0$

$\Rightarrow$ $\mathcal{R}$ is conserved on superhorizon scales $k \ll aH$!

The spectrum of scalar curvature perturbations does not change once the fluctuation has crossed the Hubble horizon!

$\Rightarrow$ Inflation gives a prediction for correlation functions constructed from $\mathcal{R}$.

Two point correlation function and power spectrum

Def.:

- Two point correlation function in position space:

$$\xi_{\mathcal{R}}(\vec{r}) \equiv \langle \mathcal{R}(\vec{x}) \, \mathcal{R}(\vec{x} + \vec{r}) \rangle$$

- $\langle \dots \rangle$ denotes the ensemble average

Relation to Fourier - amplitudes $\mathcal{R}_k$:

$$\mathcal{R}_k \equiv \int d^3x \;\; \mathcal{R}(\vec{x}) \, e^{-i \vec{k} \vec{x}}$$

Then :

$$\langle \mathcal{R}_k \mathcal{R}_{k'} \rangle$$

$$= \langle \int d^3x \ \mathcal{R}(\vec{x})\, e^{-i\vec{k}\vec{x}} \cdot \int d^3y \ \mathcal{R}(\vec{y})\, e^{-i\vec{k}'\vec{y}} \rangle$$

· change variable $\vec{y} = \vec{x} + \vec{r}$ with $\vec{x}$ fixed

$$= \int d^3x \ e^{-i\vec{k}\vec{x}} \int d^3\vec{r} \ \langle \mathcal{R}(\vec{x})\, \mathcal{R}(\vec{x}+\vec{r}) \rangle \, e^{-i\vec{k}'(\vec{x}+\vec{r})}$$

$$= \int d^3x \ e^{-i(\vec{k}+\vec{k}')\vec{x}} \int d^3\vec{r} \ \xi_{\mathcal{R}}(\vec{r})\, e^{-i\vec{k}'\vec{r}}$$

note :

$$\int d^3x \ e^{-i(\vec{k}+\vec{k}')\vec{x}} = (2\pi)^3 \ \delta^3(\vec{k}+\vec{k}')$$

gives :

$$\langle \mathcal{R}_k \mathcal{R}_{k'} \rangle = (2\pi)^3 \ \delta^3(\vec{k}+\vec{k}') \int d^3\vec{r} \ \xi_{\mathcal{R}}(\vec{r})\, e^{-i\vec{k}'\vec{r}'}$$

Comparison with summary where

$$\langle \mathcal{R}_k \mathcal{R}_{k'} \rangle = (2\pi)^3 \ \delta^3(\vec{k}+\vec{k}') \ P_{\mathcal{R}}(\vec{k}')$$

Shows : the power spectrum $P_{\mathcal{R}}(k)$ is the Fourier transform of the position space correlation function $\xi_{\mathcal{R}}(\vec{r})$

$$P_{\mathcal{R}}(k) = \int d^3r \ \xi_{\mathcal{R}}(\vec{r})\, e^{-i\vec{k}\vec{r}}$$

□

Based on $P_{\mathcal{R}}(k)$ we define:

- the dimensionless scalar power spectrum

$$\Delta_S^2 \equiv \Delta_{\mathcal{R}}^2 = \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k)$$

- the scalar spectral index

$$n_S - 1 = \frac{d \ln \Delta_S^2}{d \ln k}$$

- the running of the scalar spectral index

$$\alpha_S = \frac{d n_S}{d \ln k}$$

remarks:

- based on these definitions one can parametrise $\Delta_S^2(k)$ as

$$\Delta_S^2(k) = A_S(k_*) \left(\frac{k}{k_*} \right)^{n_S(k_*) - 1 + \frac{1}{2} \alpha_S(k_*) \ln k/k_*}$$

  - $A_S(k_*)$: normalisation factor
  - k_* : reference or pivot scale.

- if the fluctuation spectrum is gaussian, all information is contained in the two-point-correlation function.

- multi-field inflation models predict "non-Gaussianities" in the fluctuation spectrum. This becomes visible in higher order correlation functions.

Computing the scalar power spectrum

goal: compute the quantum mechanical fluctuations created during inflation

Strategy:

1) Expand the action underlying single-field slow-roll inflation to second order in $\mathcal{R}$.

2) Derive the e.o.m. for $\mathcal{R}$ and show that they are of the form of an harmonic oscillator

3) Solve the mode equation for $\mathcal{R}$
(uses slow-roll approximation / de Sitter background)

4) Promote $\mathcal{R}$ to a quantum field and perform a canonical quantization

5) Define the ground state by matching the asymptotics to flat Minkowski space

6) Compute the power spectrum at horizon crossing

6.2 Interlude : Quantum mechanics of the harmonic oscillator

- This short review summarises the background entering into the quantization of the scalar density fluctuations $\mathcal{R}$.

Classical action of a harmonic oscillator (HO) with time-dependent frequency $\omega(t)$:

$$S = \int dt \left(\frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega^2(t)\, x^2 \right) = \int dt\, L$$

- we set the particle mass $m = 1$ to simplify notation

Classical equations of motion:

$$\frac{\delta S}{\delta x} = 0 \quad \Rightarrow \quad \ddot{x} + \omega^2(t)\, x = 0$$

<u>Canonical quantization</u>

- the quantization of the system proceeds in a standard way
- construct momentum conjugate to x:

$$p \equiv \frac{dL}{d\dot{x}} \quad \rightarrow \quad p = \dot{x}$$

- promote x, p to quantum operators $\hat{x}, \hat{p}$ and impose the canonical commutator relation

$$[\hat{x}, \hat{p}] = i\hbar$$

HO: $$[\hat{x}(t), \dot{\hat{x}}(t)] = i\hbar$$

· Heisenberg picture : operators are time-dependent
states are time-independent

Expand $\hat{x}$ in creation operator $\hat{a}^+$ and annihilation operator $\hat{a}$

$$\hat{x} = v(t)\,\hat{a} + v^*(t)\,\hat{a}^+ \qquad (1)$$

the complex mode-function $v(t)$ satisfies the classical e.o.m

$$\ddot{v} + \omega^2(t)\,v = 0$$

Construction of $\hat{a}^+, \hat{a}$:

· The canonical commutation relation of $\hat{x}, \hat{p}$ induces a commutation relation of $\hat{a}^+$ with $\hat{a}$:

$$[\hat{x}, \dot{\hat{x}}] \overset{!}{=} i\hbar$$

$$= [v\hat{a} + v^*\hat{a}^+, \dot{v}\hat{a} + \dot{v}^*\hat{a}^+]$$

$$= v^*\dot{v}\,[\hat{a}^+, \hat{a}] + v\dot{v}^*\,[\hat{a}, \hat{a}^+]$$

$$= (v\dot{v}^* - v^*\dot{v})\,[\hat{a}, \hat{a}^+]$$

$$= i\hbar\,\langle v, v\rangle\,[\hat{a}, \hat{a}^+]$$

with the scalar product

$$\langle v, w\rangle \equiv \frac{i}{\hbar}\left(v^*\,\partial_t w - (\partial_t v^*)\,w\right)$$

i.e. :

$$\langle v, v\rangle = \frac{i}{\hbar}\left(v^*\dot{v} - \dot{v}^* v\right)$$

Normalizing the mode function to $\langle v, v \rangle = 1$ yields the canonical commutator for creation and annihilation operators

$$[\hat{a}, \hat{a}^+] = 1$$

$\hat{a}$, $\hat{a}^+$ can then be constructed from the mode function

$$\hat{a} = \langle v, \hat{x} \rangle \quad , \quad \hat{a}^+ = \langle v^*, \hat{x} \rangle$$

Given $\hat{a}$, $\hat{a}^+$ the Fock space of the theory is constructed in a standard way:

- Fock vacuum:

$$\hat{a} \, |0\rangle = 0$$

  state annihilated by lowering operator $\hat{a}$

- Excited states are created by acting with the creation operator

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^+\right)^n |0\rangle \qquad n \geq 0$$

- These are eigenstates of the number operator $\hat{N} \equiv \hat{a}^+ \hat{a}$ with eigenvalue n:

$$\hat{N} \, |n\rangle = n \, |n\rangle$$

Problem:

- keep solution $x(t)$ fixed and change mode function $v(t)$

$\Rightarrow$ changes the annihilation operator $\hat{a} = \langle v, \hat{x} \rangle$

$\Rightarrow$ changes the vacuum state $|0\rangle$

conclusion:

- for the HO with time-dependent frequency and also for quantum field theory in curved spacetime there is no unique vacuum state!

Solution in the case of the HO:

- consider the limit of HO with constant frequency $\omega(t) = \omega$
- in this case there is a distinguished vacuum state $|0\rangle$ which minimizes the Hamiltonian

  ($\Rightarrow$ implicit additional assumption in the usual QM treatment)

Evaluate $\hat{H}$ for a general mode function $v(t)$:

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2$$

$$= \frac{1}{2}\Big[(\dot{v}^2 + \omega^2 v^2)\,\hat{a}\,\hat{a} + (\dot{v}^2 + \omega^2 v^2)^*\,\hat{a}^+\hat{a}^+$$

$$+ (|\dot{v}|^2 + \omega^2|v|^2)(\hat{a}\,\hat{a}^+ + \hat{a}^+\hat{a})\Big]$$

determine action of $\hat{H}$ on the vacuum

- uses: $\hat{a}\,|0\rangle = 0$, $[\hat{a}, \hat{a}^+] = 1$

$$\hat{H}\,|0\rangle = \frac{1}{2}(\dot{v}^2 + \omega^2 v^2)^*\,\hat{a}^+\hat{a}^+\,|0\rangle$$

$$+ \frac{1}{2}(|\dot{v}|^2 + \omega^2|v|^2)\,|0\rangle$$

Condition that $|0\rangle$ is an eigenstate of $\hat{H}$ implies

$$\dot{v}^2 + \omega^2 v^2 = 0 \quad \rightarrow \quad \dot{v} = \pm i \omega v \tag{3}$$

- sign ambiguity from taking the square-root

$\Rightarrow$ fix by normalization condition $\langle v, v\rangle = 1$

$$\langle v, v\rangle = \frac{i}{\hbar}\left(v^* \dot{v} - \dot{v}^* v\right)$$

$$= \frac{i}{\hbar}\left(\pm i \omega\right) 2 |v|^2$$

$$= \mp \frac{2\omega}{\hbar} |v|^2$$

positivity selects the lower sign

$\Rightarrow$ mode function follows from the differential equation

$$\dot{v} = -i \omega v$$

Solution, including the correct normalization such that $\langle v, v\rangle = 1$

$$v(t) = \sqrt{\frac{\hbar}{2\omega}}\, e^{-i\omega t}$$

The mode function determines the "positive frequency solutions"

Evaluate $\hat{H}$ for this mode function:

- prefactors multiplying $\hat{a}\hat{a}$, $\hat{a}^+\hat{a}^+$ terms cancel by virtue of (3)

- $|\dot{v}|^2 = \frac{\hbar}{2}\omega \quad , \quad \omega^2 |v|^2 = \frac{\hbar}{2}\omega$

gives :

$$\hat{H} = \frac{1}{2} (\hbar\omega) (\hat{a} \hat{a}^+ + \hat{a} \hat{a}^+ + 1)$$

$$= \hbar\omega (\hat{N} + 1/2)$$

$\Rightarrow$ ground state energy $E = \frac{1}{2} \hbar \omega$

Since $\hat{N}$ is positive semi-definite, $|0\rangle$ is indeed the state of minimal energy

Zero point fluctuations

- ultimately, we are interested in vacuum fluctuations in the ground state, i.e. $\langle 0| \, |\hat{x}|^2 \, |0\rangle$

Result: these are determined by the mode function

$$\langle |\hat{x}|^2 \rangle = | v(\omega, t) |^2$$

This can be shown in a standard way using the properties of the creation and annihilation operators:

$$\langle |\hat{x}|^2 \rangle = \langle 0| \, \hat{x}^+ \hat{x} \, |0\rangle$$

$$= \langle 0| (v^* a^+ + v \hat{a})(v \hat{a} + v^* \hat{a}^+) |0\rangle$$

$$= |v|^2 \langle 0| \, \hat{a} \hat{a}^+ \, |0\rangle$$

$$= |v|^2 \langle 0| \, [\hat{a}, \hat{a}^+] \, |0\rangle$$

$$= |v|^2$$

Example : HO :

$$\langle |\hat{x}|^2 \rangle = \frac{1}{2} \hbar \omega$$

- standard QM - result

This completes the QM background required for computing the fluctuation spectrum in cosmology.

6.3 Quantum fluctuations in de Sitter space

- In the following, we focus on scalar curvature perturbations. The case of tensor perturbations encoded in h_{ij} follows the same path and we will quote the corresponding results from the literature.

Step 1: <u>The e.o.m. for scalar perturbations are equivalent to the ones of a harmonic oscillator with time-dependent frequency</u>

Starting point: action for single-field inflation

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(R - (\partial_\mu \phi)(\partial^\mu \phi) - 2V(\phi) \right) \qquad (4)$$

Parametrise fluctuations:

- we employ comoving gauge and neglect tensor fluctuations (These do not couple to $\mathcal{R}$ at the linear level)

Ansatz for perturbed FRW metric:

$$\delta \phi = 0$$

$$g_{ij} = a^2 (1 - 2\mathcal{R}) \delta_{ij}$$

where g_{ij} is the spatial part of the spacetime metric expand action (4) to second order in fluctuations $\mathcal{R}$:

$$S^{(2)} = \frac{1}{2} \int d^4x \; a^3 \frac{\dot{\phi}^2}{H^2} \left[\dot{\mathcal{R}}^2 - a^{-2} (\partial_i \mathcal{R})^2 \right] \qquad (5)$$

- the dot denotes the derivative w.r.t. cosmic time t

note that this computation is quite involved (exercise 5 of this weeks problem set is for hardcore theorists)

- Redefine fluctuation fields introducing the Mukhanov variable

$$v \equiv z \, \mathcal{R} \qquad\qquad z^2 = a^2 \frac{\dot{\phi}^2}{H^2} = 2 a^2 \varepsilon$$

- go from cosmic time t to conformal time τ, $d\tau = a^{-1} dt$

The action (5) then becomes:

$$S^{(2)} = \frac{1}{2} \int d\tau \, d^3x \left[(v')^2 - (\partial_i v)^2 + \frac{z''}{z} v^2 \right]$$

- primes are derivatives w.r.t. comoving time τ

equations of motion from variational principle:

$$\frac{\delta S^{(2)}}{\partial v} = 0 \quad \Rightarrow \quad v'' - \delta^{ij} \partial_i \partial_j v - \frac{z''}{z} v = 0$$

derive e.o.m. for Fourier-amplitudes by substituting

$$v(\tau, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \, v_k(\tau) \, e^{i \vec{k} \vec{x}}$$

yields e.o.m. for Fourier - amplitudes:

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0 \qquad (6)$$

- eom of harmonic oscillator with time-dependent frequency $\omega^2 = k^2 - \frac{z''}{z}$
- hard to solve since z''/z includes the dynamics of the background.

Step 2: Quantization and choice of vacuum

- we follow the example of the HO

promote the Fourier - components to quantum operators and introduce creation and annihilation operators:

$$v_k \rightarrow \hat{v}_k = v_k(\eta)\, \hat{a}_k + v_{-k}^*(\eta)\, \hat{a}_{-k}^+$$

where

$$[\hat{a}_k, \hat{a}_{k'}^+] = (2\pi)^3\, \delta^3(\vec{k} - \vec{k}')$$

and normalization

$$\langle v_k, v_k \rangle \equiv \frac{i}{\hbar}\left(v_k^* v_k' - v_k^{*\prime} v_k\right) = 1 \qquad (7a)$$

note:

- The normalization condition provides one of the boundary conditions for the solution of eq. (6).

Second boundary condition is fixed by the choice of vacuum $|0\rangle$

$\Rightarrow$ conventionally one chooses the Bunch - Davis vacuum

i.e. the Minkowski vacuum seen by a comoving observer

in the far past, $\tau \to -\infty$, $|k\tau| \gg 1$

In this limit $z''/z \to 0$ and (6) reduces to

$$v_k'' + k^2 v_k = 0$$

$\Rightarrow$ HO with time - independent frequency.

From the positive - frequency spectrum of the HO, we obtain the second initial condition:

$$\lim_{\tau \to -\infty} v_k = \frac{1}{\sqrt{2k}} e^{-ik\tau} \qquad (7b)$$

The boundary conditions (7a), (7b) fix the mode function completely.

<u>Step 3: Solve the mode equation</u>

- need the dependence of z''/z on conformal time τ

simplification:

- work in the de Sitter limit where

$$a = -\frac{1}{\tau} \qquad \text{(see Table of cosmic evolution)}$$

$$\varepsilon = \text{const} \to 0$$

Then :

$$z = \sqrt{2} \left(- \frac{1}{\tau} \right) \varepsilon$$

$$z'' = \sqrt{2} \cdot \left(- \frac{2}{\tau^3} \right) \varepsilon$$

$$\frac{z''}{z} = \frac{2}{\tau^2}$$

Thus the mode equation on the de Sitter background is

$$v_k'' + \left(k^2 - \frac{2}{\tau^2} \right) v_k = 0 \qquad (8)$$

remarks :

- The result derived on the de Sitter background holds more generally, in particular in the slow-roll approximation

Exact solution of eom :

$$v_k = a_1 \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau} \right) + a_2 \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau} \right)$$

- This result can be verified by substituting the solution back into (8).

Boundary conditions fix a_1 and a_2 :

- asymptotic behavior (7b) $\Rightarrow$ $a_2 = 0$
- normalization (7a) $\Rightarrow$ $a_1 = 1$

Thus we arrive at the mode function for the Bunch-Davis vacuum:

$$v_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right)$$

<u>Step 4: Scalar power spectrum on quasi-de Sitter space</u>

- on de Sitter space: consider power spectrum of $\hat{\Psi}_k = a^{-1} \hat{v}_k$:
- from HO the power spectrum is encoded in the mode function

$$\langle \hat{\Psi}_k \hat{\Psi}_{k'} \rangle = (2\pi)^3 \delta(\vec{k} + \vec{k}') \frac{|v_k(\tau)|^2}{a^2}$$

$$= (2\pi)^3 \delta(\vec{k} + \vec{k}') \frac{1}{2k} \left(1 + \frac{1}{k^2 \tau^2}\right) \cdot \frac{1}{a^2}$$

$$= (2\pi)^3 \delta(\vec{k} + \vec{k}') \frac{H^2}{2k^3} \left(1 + k^2 \tau^2\right)$$

note that on de Sitter:

$$\tau = \int \frac{1}{a} dt$$

$$= \int e^{-Ht} dt$$

$$= \frac{1}{H} e^{-Ht}$$

$$= \frac{1}{Ha}$$

Thus $H = \dfrac{1}{\tau a}$

which is used in the last step.

Thus, on superhorizon scales $|k\tau| \ll 1$:

$$\langle \hat{\Psi}_k \hat{\Psi}_{k'} \rangle \Rightarrow (2\pi)^3 \delta^3(\vec{k} + \vec{k}') \frac{H^2}{2k^3}$$

equivalently

$$P_\Psi(k) = \frac{H^2}{2k^3} \qquad \Delta_\Psi^2 = \left(\frac{H}{2\pi}\right)^2$$

On quasi-de Sitter space, the power spectrum of $\mathcal{R}$ is obtained by exploiting $\hat{\mathcal{R}} = \frac{H}{\dot{\phi}} \hat{\Psi}$

- evaluating the two-point function at horizon crossing

$$\langle \hat{\mathcal{R}}_k \hat{\mathcal{R}}_{k'} \rangle = (2\pi)^3 \delta(\vec{k} + \vec{k}') \frac{H_*^2}{2k^3} \cdot \frac{H_*^2}{\dot{\phi}_*^2}$$

gives

$$\boxed{P_{\mathcal{R}}(k) = \frac{H_*^2}{2k^3} \cdot \frac{H_*^2}{\dot{\phi}_*^2} \qquad \Delta_s^2(k) = \frac{H_*^2}{(2\pi)^2} \frac{H_*^2}{\dot{\phi}_*^2}}$$

This is the central result of the derivation

analogously one obtains for the tensor fluctuations

$$\Delta_t^2 = \frac{2}{\pi^2} H^2$$

The scalar-to-tensor ratio is

$$r = \frac{\Delta_t^2}{\Delta_s^2} = 16 \frac{\dot{\phi}_*^2}{H_*^2} \approx 16\, \varepsilon$$

6.4. Predictions for slow-roll inflation:

Recall the definition of the slow-roll parameters

$$\varepsilon = -\frac{d \ln H}{d N} \qquad \eta \equiv \varepsilon - \frac{1}{2} \frac{d \ln \varepsilon}{d N}$$

goal:

- compute spectral indices in terms of ε, η

Evaluate:

$$n_S - 1 = \frac{d \ln \Delta_S^2}{d \ln k}$$

Split r.h.s. according to:

$$\frac{d \ln \Delta_S^2}{d \ln k} = \frac{d \ln \Delta_S^2}{d N} \cdot \frac{d N}{d \ln k}$$

Evaluate 1. factor:

$$\frac{d \ln \Delta_S^2}{d N} = 2 \frac{d \ln H}{d N} - \frac{d \ln \varepsilon}{d N}$$

$$= -2\varepsilon - (-2)(\eta - \varepsilon)$$

$$= 2\eta - 4\varepsilon$$

Evaluate second factor using horizon crossing condition

$$k = a H \quad \rightarrow \quad \ln k = N + \ln H + \text{const}$$

Hence :

$$\frac{dN}{d\ln k} = \left[\frac{d \ln k}{dN} \right]^{-1} = \left[1 + \frac{d\ln H}{dN} \right]^{-1} \simeq 1 + \varepsilon$$

Thus :

$$n_s - 1 = 2\eta_* - 4\varepsilon_*$$

analogously :

$$n_t = -2\varepsilon_*$$

Substituting the slow-roll parameters

$$\varepsilon \simeq \varepsilon_V \ , \quad \eta \simeq \eta_V - \varepsilon_V$$

gives the results quoted in the summary.

Central conclusion :

Any deviation from a perfectly scale-invariant power spectrum ($n_s = 1$, $n_t = 0$) is an indirect probe of the inflationary dynamics quantified in the slow-roll parameters ε and η.