7. Big bang nucleosynthesis (BBN)

observation : the baryonic matter of the universe consists of

- hydrogen ≃ 75 %
- helium ≃ 25 %

remark :

- we are quoting mass-fractions, i.e. contributions to S_B
- other elements exist, but give a negligible contribution

puzzle :

- The observed helium fraction is much bigger than what can reasonably be obtained from nuclear fusion within stars.

success of big bang nucleosynthesis :

- explains the observed mass-fractions of light elements created in the early universe
  (happening approx 1 sec. after the big bang singularity)

goal :

- understand where this prediction comes from.

Tool :

- statistical physics

Connecting milestones in the history of the universe

- this serves as the connection to the other parts of the lectures :

initial singularity

$\downarrow$

inflation (solves : flatness problem, horizon problem, relic problem
predicts : correct seeds for structure formation)

$\downarrow$

reheating (decay of inflaton particles)

$\downarrow$

hot, dense universe, particles are in thermal equilibrium

$\downarrow$ universe cools $\Leftrightarrow$ phase transitions

formation of p, n (1 GeV)

$\downarrow$

formation of light elements (1 MeV)

$\downarrow$

formation of CMB (0 1 eV)

remark :

- possibly there is also a phase - transition related to the production of dark matter (e.g. 10 GeV)
- Our focus is on the last 4 epochs.

7. 1. Particles in thermal equilibrium

- thermal equilibrium requires frequent interactions of particles so that the temperature of the cosmic plasma is uniform

rule of thumb :

- if the reaction rate of a process is much faster than the Hubble parameter particles retain thermal equilibrium
- if the reaction rates drop below the Hubble parameter the equilibrium can no longer be retained and the constituents freeze out from the cosmic plasma.

Description of particles in thermal equilibrium

- universe is populated by two types of particles : bosons and fermions
- given a system in thermal equilibrium with temperature T the energy distribution among the particles follows :

bosons : Bose - Einstein - distribution

$$f(E,T) = \frac{1}{e^{(E-\mu)/kT} - 1}$$

fermions : Fermi - Dirac distribution

$$f(E,T) = \frac{1}{e^{(E-\mu)/kT} + 1}$$

Here :

- $E^2 = \vec{p}^{\,2} + m^2$ is the energy of the particle
- μ denotes a chemical potential for the particle species
- k_B is Boltzmann's constant (we set $k_B = 1$ in the rest of the notes)

The difference in the distributions come from the Pauli-blocking, saying that it is impossible that two fermions occupy the same state (i.e. have identical quantum numbers).

We introduce g_i denoting the degeneracy of states for a given particle species.

Examples :

- photons have two polarizations yielding $g_\gamma = 2$
- electrons can have spin up / down $g_{e^-} = 2$

The number density of a given particle species is obtained by integrating the energy-distribution over phase space

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} \, f_i(E(\vec{p}), T)$$

- The factor $(2\pi\hbar)^{-3}$ (where we set $\hbar = 1$)

  is the "volume" of a single state in phase space

Based on the distributions, we also define the energy density and pressure density of a species:

$$\rho_i = \frac{g_i}{(2\pi)^3} \int d^3\vec{p} \quad E(\vec{p}) \quad f_i(E(\vec{p}), T)$$

$$P_i = \frac{g_i}{(2\pi)^3} \int d^3\vec{p} \quad \frac{|\vec{p}|^2}{3E} \quad f_i(E(\vec{p}), T)$$

In cosmology particles are distinguished into:

- relativistic particles and non-relativistic particles

In these limits the integrals can be evaluated in closed form: (also see exercise 2)

- relativistic case: dilute system

  $T \gg m, \quad |\mu| \ll T$

  in this case the mass becomes negligible compared to the momentum contribution of the total energy

$$E = \sqrt{\vec{p}^2 + m^2} \simeq |\vec{p}|$$

in this case:

$$n = \begin{cases} \frac{3}{4\pi^2}\, \zeta(3)\, g\, T^3 & \text{fermions} \\ \frac{1}{\pi^2}\, \zeta(3)\, g\, T^3 & \text{bosons} \end{cases}$$

here $\zeta(z) \equiv \sum_{n=1}^{\infty} n^{-z}$ is the Riemann zeta-function with $\zeta(3) \simeq 1.202$

analogously:

$$\rho = \begin{cases} \frac{7}{8}\, \frac{\pi^2}{30}\, g\, T^4 & \text{fermions} \\ \frac{\pi^2}{30}\, g\, T^4 & \text{bosons} \end{cases}$$

and (since the particles are relativistic)

$$p = 1/3\, \rho$$

<u>The non-relativistic limit</u>

- in this limit $T \ll m$:
- kin. energy much smaller than mass energy:

$$E \simeq m + \frac{|\vec{p}|^2}{2m}$$

- since E/T is large the exponentials may be approximated

$$f_i = \frac{1}{e^{(E-\mu)/T} \pm 1} \simeq e^{\mu/T}\, e^{-m/T}\, e^{-\frac{|\vec{p}|^2}{2mT}}$$

- the $\pm$ in the denominator can be neglected, hence there is no difference between bosons and fermions in the non-relativistic limit:

evaluating the number, energy-, and pressure density yields (all integrals are Gaussian)

$$n_i = g_i \left(\frac{m T}{2\pi} \right)^{3/2} e^{-\frac{m-\mu}{T}}$$

$$\rho_i = n_i \left(m + \frac{3}{2} T \right)$$

$$P_i = n_i T \ll \rho$$

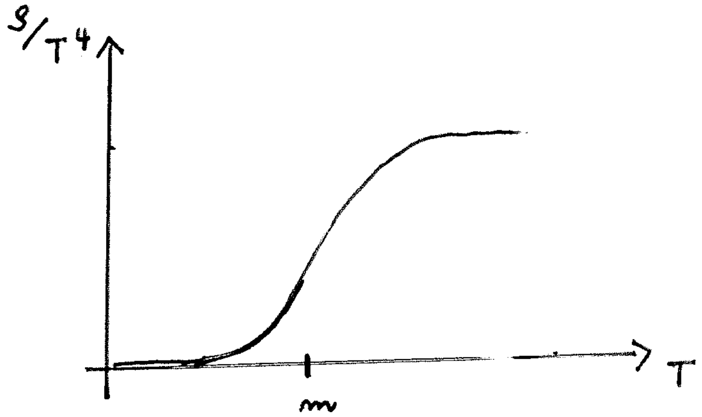
remarks:

- The structure of ρ_i is intuitive: it is the total mass of the particles $n_i m$ plus $\frac{3}{2} n_i k T$ which is the average kin. energy of a gas of non-interacting particles without internal degrees of freedom moving in a 3-dimensional box.

- in the "intermediate" regimes where the relativistic / non-relativistic approximations are not valid, the integrals can be evaluated numerically only. In cosmology these intermediate regimes are typically neglected: a species is either relativistic or non-relativistic.

Illustration :

- energy density as a function of temperature



for

$m \ll T$: ρ/T^4 is constant

(thermal equilibrium : particle / antiparticle annihilation compensated by pair production)

$m \gg T$: exponential decay of energy density ρ_i

(energy is insufficient for pair production).

- in cosmology, one considers various notions of equilibrium:
  - kinetic equilibrium
    - a specific particle species i follows the energy distribution f_i with chemical potential μ_i and temperature T_i

      T_i can be different from the temperature of the cosmic plasma (typically defined by the temperature of the photon path).

      example: neutrinos' ν_e after electron - positron annihilation
  - thermal equilibrium
    - all species have the same temperature $T = T_i \;\; \forall i$
- chemical equilibrium
  - the process of creation / annihilation of species is described by reaction formulas of the type

$$i + j \leftrightarrow k + l$$

examples:

$$p + \bar{\nu} \leftrightarrow n + e^{+}$$

$$p + e^{-} \leftrightarrow n + \nu$$

$$n \leftrightarrow p + e^{-} + \bar{\nu}$$

The reaction is in chemical equilibrium if

$$\mu_i + \mu_j = \mu_k + \mu_l$$

Application :

· differences in number densities of particles and antiparticles

if the particle has chemical potential μ

the anti-particle has chemical potential $(-\mu)$

Evaluate $n - \bar{n}$ in the relativistic limit for fermions

- approximate $E = |\vec{p}|$
- carry out integral over angular variables :

$$n - \bar{n} = \frac{g}{(2\pi)^3} \int_0^\infty (4\pi p^2)\, dp \left(\frac{1}{e^{(p-\mu)/T} + 1} - \frac{1}{e^{(p+\mu)/T} + 1} \right)$$

Evaluation of integrals uses

$$\int_0^\infty dx\, x^2 \frac{1}{e^{(x-\mu)/T} + 1} = -2T^3 \, Li_3(-e^{\mu/T})$$

and the identity for polylogarithm :

$$Li_3(z) - Li_3(1/z) = -\frac{1}{6} \log^3(-z) - \frac{\pi^2}{6} \log(-z)$$

Then :

$$n - \bar{n} = \frac{g}{(2\pi)^3} (4\pi)(-2T^3) \left(-\frac{1}{6} \left(\frac{\mu}{T} \right)^3 - \frac{\pi^2}{6} \frac{\mu}{T} \right)$$

$$= \frac{g}{6\pi^2} T^3 \left(\pi^2 \frac{\mu}{T} + \left(\frac{\mu}{T} \right)^3 \right)$$

- allows to determine the net-particle number

if there is a non-zero chemical potential $\mu \neq 0$.

7.2. The Boltzmann equation: physics out of equilibrium

goal: understand dynamical changes in the number density of a particle species

Basis: reversible process involving species 1, 2, 3, 4 interacting as

$$1 + 2 \leftrightarrow 3 + 4$$

particles 1 + 2 can annihilate to produce particles 3 + 4 and vice versa. Example $e^+ + e^- \leftrightarrow \gamma\gamma$

Idea of the Boltzmann equation:

The rate of change in the abundance of a particle is the difference between the rates of producing it and annihilating them.

This is made precise by the Boltzmann equation in an expanding universe:

$$a^{-3} \frac{d}{dt}(n_1 a^3)$$

$$= \int \frac{d^3 p_1}{(2\pi)^3 \, 2E_1} \int \frac{d^3 p_2}{(2\pi)^3 \, 2E_2} \int \frac{d^3 p_3}{(2\pi)^3 \, 2E_3} \int \frac{d^3 p_4}{(2\pi)^3 \, 2E_4}$$

$$\cdot (2\pi)^4 \, \delta^3(p_1 + p_2 - p_3 - p_4)\, \delta(E_1 + E_2 - E_3 - E_4)\, |M|^2$$

$$\{ f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4) \}$$

Explanation :

- limit where the reaction terms on the r.h.s. are zero :

$$a^{-3} \frac{d}{dt} (n_1 a^3) = 0$$

The total number of particles in the volume a^3 is conserved

- interactions are encoded in the r.h.s. :

The integrals $\int \frac{d^3 p_i}{(2\pi)^3 \, 2 E_i}$ integrate over the entire phase

space of the particle species " i "

Note : factor $2 E_i$ is required to make the integrals

Lorentz invariant. It arises from the onshell condition

$$\int d^3 \vec{p} \int dE \;\; \delta (E^2 - \vec{p}^{\,2} - m^2)$$

$$= \int d^3 \vec{p} \int dE \;\; \frac{\delta (E - \sqrt{\vec{p}^{\,2} + m^2})}{2 E}$$

using the property of the δ - distribution

$$\delta (f(x)) = \frac{1}{| f'(x) |} \, \delta (x - x_i) , \quad f(x_i) = 0$$

and

$$f'(E) = 2 E$$

- second line :

δ - distributions implement conservation of the total energy

and 3 - momentum.

- $|\mathcal{M}|^2$: amplitude of the process

  Typically, this is a particle physics input,

  example : $e^+ e^- \to \gamma\gamma$ $\quad |\mathcal{M}|^2 \propto \alpha^2$

  α^2 : fine - structure constant (cf. Peskin & Schröder)

last line : $\{\dots\}$ contains the kinetic factors :

- the increase of n_1 is proportional to the number of species $f_3 \cdot f_4$

  correction terms :

  $(1 \pm f_i)$: + for bosons implements bose enhancement

  − for fermions : blocking due to Pauli principle

- decrease of n_1 is proportional to $f_1 \cdot f_2$

  - again correction factors apply as before

summary :

- The Boltzmann equation is a complicated non - linear integro - differential equation (complicated to solve).

Typical approximations in cosmology :

- typically : non - equilibrium situations appear if $T < E - \mu$

  $\Rightarrow$ neglect ∓ 1 in Bose - Einstein / Fermi - Dirac distribution

$$f(E) \simeq e^{\mu/T} e^{-E/T}$$

- the system is dilute so that Bose enhancement / Pauli blocking is negligible

Using energy conservation the kinematical terms simplify:

$$\{ f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4) \}$$

$$\rightarrow \quad e^{-(E_1 + E_2)/T} \{ e^{(\mu_3 + \mu_4)/T} - e^{(\mu_1 + \mu_2)/T} \}$$

The chemical potentials can be expressed via the number densities:

- recall (consistent with previous approximation)

$$n_i = g_i \, e^{\mu_i/T} \int \frac{d^3p}{(2\pi)^3} \, e^{-E_i/T}$$

introduce the species - dependent equilibrium number density:

$$n_i^{(0)} = g_i \int \frac{d^3p}{(2\pi)^3} \, e^{-E_i/T}$$

gives

$$e^{\mu_i/T} = \frac{n_i}{n_i^{(0)}}$$

Thus

$$e^{-(E_1 + E_2)/T} \{ e^{(\mu_3 + \mu_4)/T} - e^{(\mu_1 + \mu_2)/T} \}$$

$$= e^{-(E_1 + E_2)/T} \left\{ \frac{n_3 \, n_4}{n_3^{(0)} \, n_4^{(0)}} - \frac{n_1 \, n_2}{n_1^{(0)} \, n_2^{(0)}} \right\}$$

Finally: define the thermally averaged cross-section

$$\langle \sigma v \rangle \equiv \frac{1}{n_1^{(0)} n_2^{(0)}} \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \int \frac{d^3 p_2}{(2\pi)^3 2E_2} \int \frac{d^3 p_3}{(2\pi)^3 2E_3} \int \frac{d^3 p_4}{(2\pi)^3 2E_4}$$

$$\cdot\, e^{-(E_1+E_2)/T} \cdot (2\pi)^4 \, \delta^3(p_1 + p_2 - p_3 - p_4)\, \delta(E_1 + E_2 - E_3 - E_4)$$

$$\cdot\, |\mathcal{M}|^2$$

With these definitions / approximations the Boltzmann-equation is turned into a set of ordinary differential equations:

$$a^{-3} \frac{d}{dt}(n_1 a^3) = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\}$$

allows to track abundances of particles out of thermal equilibrium.

Observation:

lhs: typical order $n_1 H \sim n_1/t$

rhs: $n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle$

$\Rightarrow$ $n_2^{(0)} \langle \sigma v \rangle$ gives the reaction rate

If the reaction rate is much larger than the expansion rate, the terms on the r.h.s. will be much larger than the l.h.s.

$\Rightarrow$ equality requires terms in brackets to cancel:

Thus l.h.s = r.h.s. requires

$$\frac{n_3\, n_4}{n_3^{(0)}\, n_4^{(0)}} = \frac{n_1\, n_2}{n_1^{(0)}\, n_2^{(0)}}$$

Depending on the application this equation implements

- heavy relics production $\Leftrightarrow$ chemical equilibrium
- BBN $\Leftrightarrow$ nuclear statistical equilibrium
- recombination of e^- and p $\Leftrightarrow$ Saha equation

note that all these (quite different) physical processes can be described by the same tools from statistical physics.

7.3. Application : Big Bang nucleosynthesis

goal : use equilibrium equation to understand the distribution of light elements in the early universe

H : mass : 75%

He^4 : mass : 25%

one He^4 nucleus per 12 Hydrogen atoms

approximations :

- we neglect all contributions from heavier nuclei
- deuterium in the universe is minimal (< 0.01% mass) owed to the low binding energy

Cosmic enviroment at $T = 1\,MeV$:

- $1\,MeV$ is the typical binding energy of nucleons, thus it is natural that nucleosynthesis occurs at around this scale

Cosmic inventory :

- relativistic particles in equilibrium :

  photons, electrons, positrons, $e^+e^- \leftrightarrow \gamma\gamma$
- decoupled relativistic particles :

  neutrinos
- non-relativistic particles : protons / neutrons

Computing the He^4 mass - fraction

Step 1: Estimate the ratio of neutrons n_n to protons n_p

- difference in energy:

$$Q = m_n - m_p = 1.293 \text{ MeV}$$

- since the nucleons are non-relativistic, their number density as a function of temperature is
(n and p are fermions with two spin states $\rightarrow g = 2$)

$$n_p = 2 \left(\frac{m_p T}{2\pi} \right)^{3/2} e^{\frac{\mu_p - m_p}{T}}$$

$$n_n = 2 \left(\frac{m_n T}{2\pi} \right)^{3/2} e^{\frac{\mu_n - m_n}{T}}$$

note:

- mass - difference in polynomial prefactor can be neglected
- chemical potentials play no role.
(this can be estimated from studying the chemical potentials for $n + \nu_e \leftrightarrow p + e^-$)

Then:

$$\frac{n_n^{(0)}}{n_p^{(0)}} = e^{-Q/T}$$

Step 2: the time-frame of forming light elements

- there are no 4 nucleus reactions of the type

$$p + p + n + n \rightarrow He^4$$

The production of He^4 uses deuterium d via the chain:

$$p + n \rightarrow d + \gamma$$

$$d + d \rightarrow n + He^3$$

$$He^3 + d \rightarrow p + He^4$$

This leads to the deuterium bottle-neck:

- due to the low binding energy, deuterium production starts at $T \simeq 0.1$ MeV
- the reaction $d + d$ has to overcome a coulomb barrier (two positively charged nuclei repel)

  $\Rightarrow$ nucleosynthesis shuts off if nuclei have insufficient kinetic energy to overcome the barrier

  $$T \simeq 0.03 \text{ MeV}$$

$\Rightarrow$ This leaves less than one hour to form light elements!

Step 3: Tracing the neutron abundance

- relevant nuclear process relating protons and neutrons:

$$n + l \leftrightarrow p + l$$

where l is a lepton, e.g. $n + \nu \leftrightarrow p + e^-$

all leptons are light and therefore in thermal equilibrium

$$n_l \equiv n_l^{(0)}$$

- applying Boltzmann's equation to n_n:

$$a^{-3} \frac{d}{dt} (n_n a^3) = n_l^{(0)} \langle \sigma v \rangle \left\{ \frac{n_p \, n_n^{(0)}}{n_p^{(0)}} - n_n \right\} \quad (1)$$

- The thermally averaged cross-section is encoded in

$$\lambda_{np} \equiv n_l^{(0)} \langle \sigma v \rangle$$

- introduce new variables:
  - neutron mass-fraction:

$$X_n = \frac{n_n}{n_n + n_p}$$

  - new time variable $t \to x \equiv Q/T$

using that the total number of baryons is conserved

$$\frac{d}{dt} (n_n + n_p) \cdot a^3 = 0$$

eq (1) becomes a differential eq. for $X_n(x)$:

$$\frac{dX_n}{dx} = \frac{\lambda_{pn}}{H(x=1)} \cdot x \cdot \left\{ e^{-x} (1 - X_n) - X_n \right\} \qquad (2)$$

Here

$$H|_{x=1} = 1.13\, s^{-1}$$

$$\lambda_{pn} = \frac{255}{\tau_n\, x^5} \left(12 + 6x + x^2 \right)$$

- solving (2) numerically => $X_n \simeq 0.15$ for $T \gtrsim 0.1$ MeV

<u>Addendum: derivation of (2) from eq. (1)</u>

- start on l.h.s.:

$$a^{-3} \frac{d}{dt} (n_n a^3)$$

$$= a^{-3} \frac{d}{dt} \left(\frac{n_n}{n_n + n_p} \cdot (n_n + n_p) a^3 \right)$$

$$= (n_n + n_p) \frac{d}{dt} X_n$$

uses that total number of baryons is conserved:

- chain rule on l.h.s. - need to evaluate the Jacobian

$$\frac{d}{dt} X_n = \frac{dx}{dt} \frac{dX_n}{dx}$$

definition of $x = \frac{Q}{T}$ yields

$$\frac{dx}{dt} = -Q \frac{1}{T^2} \frac{dT}{dt} = -x \frac{1}{T} \frac{dT}{dt} = x H$$

in the last step we used that $T \propto a^{-1}$ so that

$$\frac{1}{T}\frac{dT}{dt} = a\left(-\frac{1}{a^2}\right)\frac{da}{dt} = -H$$

The Hubble parameter is related to the temperature via Friedmann's equations:

$$H^2 = \frac{8\pi G}{3}\rho$$

$$= \frac{8\pi G}{3}\cdot\frac{\pi^2}{30} g_* T^4$$

$$\Rightarrow \quad H = H(x=1)\,\frac{1}{x^2}$$

where $H^2(x=1) = \frac{8\pi G}{3}\frac{\pi^2}{30} g_* Q^4$

sets the time-scale;

At $T = 1\,\mathrm{MeV}$ the relativistic particles contributing to g_* are

- photons $g_\gamma = 2$
- neutrinos $g_\nu = 3\cdot 2$
- electrons / positrons: $g_{e^+} = g_{e^-} = 2$

Thus

$$g_* = \underset{\gamma}{\underset{\uparrow}{2}} + \frac{7}{8}\left(\underset{\nu}{\underset{\uparrow}{6}} + \underset{e^+}{\underset{\uparrow}{2}} + \underset{e^-}{\underset{\uparrow}{2}}\right) = 10.75$$

Then $H \simeq 1.13\,s^{-1}$ sets the time-scale

Substituting the intermediate results and writing the r.h.s. in terms of X_n and x yields

$$\frac{d X_n}{d x} = \frac{\lambda_{pn}}{H(x)\, x} \left\{ e^{-x} (1 - X_n) - X_n \right\}$$

$$= \frac{\lambda_{pn} \cdot x}{H(x=1)} \left\{ e^{-x} (1 - X_n) - X_n \right\}$$

□

4. neutron loss due to β - decay

- X_n freezes at $T = 0.1$ MeV
- deuterium production becomes effective at $T = 0.07$ MeV

⇒ neutrons decay:

$$n \rightarrow p + e^- + \bar{\nu} \qquad \tau_n = 886 \text{ sec.}$$

cooling down from $T = 0.1$ MeV to $T = 0.07$ MeV :

$$t = 132 \, s \left(\frac{0.1 \text{ MeV}}{T} \right)^2$$

Thus taking β - decay into account

$$X_n (T_{nuc}) = X_n (T = 0.1 \text{ MeV}) \, e^{- \frac{132}{886} \cdot \left(\frac{0.1}{0.07} \right)^2}$$

$$= 0.11$$

Step 5: light element abundance:

good approximations:

- He^4 is produced instantaneously at $T = 0.07$ MeV as a sufficient amount of deuterium is available
- all free neutrons are converted to He^4 (high binding energy)

Then: He^4 has two neutrons and 2 protons.

Thus its mass-fraction is

$$X_{He^4} = 2\, X_n(T_{nuc}) \simeq 0.22$$

remarks:

- this agrees very well with the observed value of primordial He^4
- pinpoints physics in the early universe

  1 sec after the Big Bang the universe should be in thermal equilibrium for nucleosynthesis to work!