Cosmology

Solutions : tutorial week 16

Exercise 1: cosmology revisited

a) 1) In thermal equilibrium the occupation number of states for Bosons is given by the Bose-Einstein distr.

$$f^{BE} = \frac{1}{e^{(E-\mu)/T} - 1}$$

while fermions obey the Fermi-Dirac statistics

$$f^{FD} = \frac{1}{e^{(E-\mu)/T} + 1}$$

Owed to the different distributions entering into the energy density

$$\rho = g_i \int \frac{d^3p}{(2\pi)^3} f_i(\vec{x}, \vec{p}) \, E(p)$$

bosonic and fermionic degrees of freedom contribute to the total energy budget with a different weight

$$g_{*,B} = g_i \quad , \quad g_{*F} = \frac{7}{8} g_i$$

2) At $T = 200\,\text{GeV}$ there are more relativistic species than at $T = 1\,\text{MeV}$

at $T \simeq 200\,\text{GeV}$ all standard model particles are relativistic giving $g_* = 28 + \frac{7}{8}\, 90 = 106.75$

at $T = 1\,MeV$ we have $g_* = 10.75$

b) translation symmetry ensures the decoupling of Fourier modes at the linearized level

c) When deriving the cosmic fluctuation spectra, the vacuum state of the universe is assumed to be the Bunch-Davies vacuum

d) The slow-roll parameters ϵ_V, η_V encode the shape of the potential. Information on the initial velocity $\dot{\phi}$ is missing. If $\dot{\phi}$ is too large, inflation may not be realized even if the potential admits regions where the slow-roll parameters are sufficiently small.

Exercise 2: Slow-roll inflation revisited

goal: analyze an inflationary model where $V = \lambda \phi^4$

a) The potential slow-roll parameters are defined as

$$\varepsilon_V = \frac{M_{Pl}^2}{2} \left(\frac{V'}{V} \right)^2$$

$$\eta_V = M_{Pl}^2 \frac{V''}{V}$$

For the specific model, one has

$$V = \lambda \phi^4 \quad , \quad V' = 4 \lambda \phi^3 \quad , \quad V'' = 12 \lambda \phi^2$$

Thus

$$\varepsilon_V = \frac{8 M_{Pl}^2}{\phi^2} \qquad \eta_V = \frac{12 M_{Pl}^2}{\phi^2} \qquad (1)$$

b) Inflation ends if either $\varepsilon_V = 1$ or $\eta_V = 1$

$$\varepsilon_V = 1 \qquad \phi_{end}^2 = 8 M_{Pl}^2$$

$$\eta_V = 1 \qquad \phi_{end}^2 = 12 M_{Pl}^2$$

The inflationary regime is for $|\phi| \gg 1 \Leftrightarrow \eta_V, \varepsilon_V \ll 1$

Thus the condition indicating the end of inflation is $\eta_V = 1$

$$\phi_{end}^2 = 12 M_{Pl}^2$$

c) The number of e-folds for $\phi_{init} > \phi_{end}$ is obtained from:

$$N = \frac{1}{M_{Pl}} \int_{\phi_{end}}^{\phi_{init}} \frac{d\phi}{\sqrt{2\varepsilon_V}}$$

substituting the slow-roll parameter gives

$$N = \frac{1}{M_{Pl}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{8 M_{Pl}^2}} \int_{\phi_{end}}^{\phi_{init}} \phi \, d\phi$$

$$= \frac{1}{8 M_{Pl}^2} \left(\phi_{init}^2 - \phi_{end}^2 \right)$$

d) We use the result c) as input. Neglecting the term ϕ_{end}^2 we get

$$N_{60} = \frac{1}{8 M_{Pl}^2} \phi_{CMB}^2 = N_{CMB}$$

$$\phi_{CMB} = \sqrt{8 \cdot 60} \, M_{Pl} = 21.9 \, M_{Pl}$$

e) Having from d)

$$\phi_{CMB}^2 = 8 M_{Pl}^2 N_{CMB}$$

the evaluation of (1) at the CMB scale is

$$\varepsilon_V \big|_{CMB} = \frac{8 M_{Pl}^2}{\phi_{CMB}^2} = \frac{1}{N_{CMB}}$$

$$\eta_V \big|_{CMB} = \frac{12 M_{Pl}^2}{\phi_{CMB}^2} = \frac{3}{2} \frac{1}{N_{CMB}}$$

f) The scalar power spectrum has $\Delta_s^2 \sim 10^{-9}$

From the lectures we have

$$\Delta_s^2 = \frac{1}{24\pi^2} \frac{V(\phi)}{M_{Pl}^4} \cdot \frac{1}{\varepsilon_V} \Big|_{\phi = \phi_{CMB}}$$

$$= \frac{1}{24\pi^2} \frac{\lambda \phi_{CMB}^4}{M_{Pl}^4} \cdot \frac{\phi_{CMB}^2}{8 M_{Pl}^2}$$

$$= \frac{1}{24\pi^2} \lambda \cdot \frac{(8 M_{Pl}^2 N_{CMB})^2}{M_{Pl}^4} \cdot \frac{8 M_{Pl}^2 N_{CMB}}{8 M_{Pl}^2}$$

$$= \frac{1}{24\pi^2} \lambda N_{CMB}^3 \cdot 64$$

Thus

$$\lambda = \Delta_s^2 \cdot \frac{24\pi^2}{64 N_{CMB}^3}$$

$$\simeq 10^{-9} \cdot \frac{24\pi^2}{64 \cdot 60^3} \simeq 1.7 \times 10^{-14}$$

g) The spectral index n_S is defined as

$$n_S = 1 + 2\eta_V - 6\varepsilon_V$$

substituting the results from e)

$$n_S = 1 + 2 \cdot \frac{3}{2} \frac{1}{N_{CMB}} - 6 \frac{1}{N_{CMB}}$$

$$= 1 - \frac{3}{N_{CMB}}$$

$$\simeq 0.95$$

where we used $N_{CMB} \simeq 60$ in the last step.

$$r = 16\,\varepsilon_V = \frac{16}{N_V}$$

$$\simeq 0.267$$

The r-ratio is by now ruled out by experiments

(Planck puts an upper bound $r \lesssim 0.11$)

see arXiv: 1502.02114

Exercise 3: non-equilibrium phases

1) Formation of hadrons

- this occurs at the confinement / deconfinement transition of QCD. Free quarks are bound into hadrons and their anti-particles, e.g.

$$u\,u\,d \rightarrow p \quad , \quad u\,d\,d \rightarrow n$$

$$\bar{u}\,\bar{u}\,\bar{d} \rightarrow \bar{p} \quad , \quad \bar{u}\,\bar{d}\,\bar{d} \rightarrow \bar{n}$$

This happens at $T \simeq 150\ MeV$

$$t \simeq 10^{-6}\ s$$

2) Leptogenesis

Hadrons and anti-hadrons annihilate into leptons and anti-leptons:

$$p + \bar{p} \rightarrow e^{+} + e^{-} \qquad etc$$

this happens at $t \sim 1\ s$

3) creation of light elements:

the key reaction is the formation of deuterium

$$n + p \rightarrow D + \gamma$$

which is processed further to ^{3}He

$T \simeq 0.1\ MeV \qquad 10\,s$ to $10^{3}\ s$

4) Formation of the CMB:

Photons are no longer energetic enough to ionize hydrogen

$$e^- + p \leftrightarrow H + \gamma$$

This happens at $T \simeq 1\,eV$ or at $t = 3.8 \cdot 10^5$ years

Exercise 4 : Tracking the neutron fraction

a) finding the initial conditions :

at $T \simeq 1$ MeV neutrons and protons are non - relativistic

Thus their number densities follow the Boltzmann distribution : ($k = 1$, $c = 1$) :

$$n_i \propto e^{-E_i/T} \propto e^{-m_i/T}$$

The ratio of two Boltzmann distributions is proportional to the energy difference :

defining : $Q \equiv m_n - m_p = 1.293$ MeV

we have

$$\frac{n_n}{n_p} = e^{-m_n/T} \, e^{m_p/T}$$

$$= e^{-Q/T}$$

Setting the initial conditions at temperature $T = 5Q$

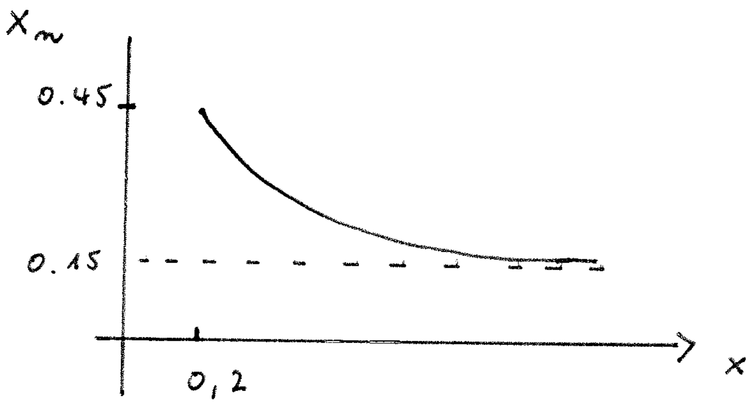
$x \equiv \frac{Q}{T} = 1/5$ we have from

$$X_n \equiv \frac{n_n}{n_n + n_p} = \frac{1}{1 + n_p/n_n}$$

$$X_n \Big|_{x_{init}} = \frac{1}{1+e^{x}} \Bigg|_{x = 1/5}$$

$$= \frac{1}{1+e^{1/5}} = 0.45$$

b) Solving the differential equation numerically yields a solution of the form:



Thus X_n indeed stabilizes to $X_n \simeq 0.15$ for $x >> 1$

c) The temperature $T = 0.1$ MeV corresponds to

$$x = \frac{Q}{T} = \frac{1.293 \text{ MeV}}{0.1 \text{ MeV}} = 12.93$$

Evaluating the numerical solution at this value gives

$$X_n \Big|_{x = 12.93} = 0.152$$

This value sets the neutron abundance for nucleosynthesis.