Cosmology 15/16

Solutions : Problem set # 1

Exercise 1 : Age of the universe :

a) Evaluate (1) at $t = t_0$:

$$H(t_0) = H_0 \quad , \quad a(t_0) = a_0$$

using standard conventions

Thus

$$1 = \Omega_\Lambda + \frac{1 - \Omega_\Lambda}{(a_0)^3}$$

must hold for any value Ω_Λ :

Thus $a_0 = 1$ satisfying the canonical normalization

b) Substitute $H = \frac{\dot{a}}{a}$ and solve for dt

$$\left(\frac{1}{H_0}\right)^2 \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \Omega_\Lambda + \frac{1 - \Omega_\Lambda}{a^3}$$

$$dt = H_0^{-1} \frac{da}{a} \left[\Omega_\Lambda + \frac{1 - \Omega_\Lambda}{a^3} \right]^{-1/2}$$

$\square$

c)

$$\int_0^{t_0} dt = H_0^{-1} \int_0^1 \frac{da}{a} \left[\Omega_\Lambda + \frac{1 - \Omega_\Lambda}{a^3} \right]^{-1/2}$$

The integral can be done analytically :

$$I = \int \frac{dx}{x} \left[\Omega + \frac{(1-\Omega)}{x^3} \right]^{-1/2}$$

$$= \frac{2}{3} \frac{1}{\sqrt{\Omega}} \log \left[x^{3/2} \Omega + \sqrt{\Omega} \sqrt{1 + (x^3 - 1)\Omega} \right]$$

limit $x \to 0$

$$I = \frac{1}{3} \frac{1}{\sqrt{\Omega}} \log (\Omega (1 - \Omega))$$

gives

$$t_0 = \left[\frac{2}{3} \frac{1}{\sqrt{\Omega}} \log \left[\Omega + \sqrt{\Omega} \right] - \frac{1}{3} \frac{1}{\sqrt{\Omega}} \log (\Omega (1 - \Omega)) \right] H_0^{-1}$$

d) age of the universe for $\Omega_\Lambda = 0$

note limits :

$$\lim_{\Omega \to 0} [\ldots] = \frac{2}{3}$$

This can also derived straightforwardly from the integral c) by first setting $\Omega_\Lambda = 0$ and subsequently integrating

Thus

$$t_0 = \frac{2}{3} \frac{0.98 \cdot 10^{10} \text{ years}}{0.72}$$

$$\simeq 9{,}07 \cdot 10^9 \text{ years}$$

Clearly this is in contradiction with the oldest objects observed in the universe. Thus a completely matter-dominated universe with $\Omega_m = 1$ is ruled out observationally

e) For $\Omega_\Lambda = 0.7$ the prefactor multiplying ~~the~~ Hubble parameter is

$$[\ldots] = 0.964$$

Thus

$$t_0 \simeq 1.31 \cdot 10^{10} \text{ years}$$

almost matches the age of the oldest objects.

Exercise 2: Galaxy rotation curves

a) From classical mechanics, we know that the gravitational field created by a spherically symmetric mass-distribution is governed solely be the mass $M(r)$ sitting <u>inside</u> the shell of radius r and creates a gravitational field equivalent to the one of the same mass situated in the center of the sphere.

To derive eq. (4) we balance the centrifugal force of a star of mass m orbiting the center at a distance r with (non-relativistic) velocity v

$$F_c = m \frac{v^2}{r}$$

with the gravitational pull of a point mass $M(r)$ sitting at the origin:

$$F_G = \frac{G\, M(r)\, m}{r^2}$$

$$F_c = F_G$$

$$m \frac{v^2}{r} = \frac{G\, M(r)\, m}{r^2}$$

$$v = \sqrt{\frac{G\, M(r)}{r}}$$

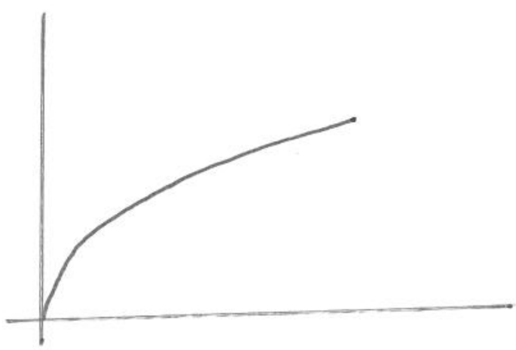
b) Given a spherically symmetric mass density $\rho(r)$ the total mass inside a sphere of radius R is obtained by integrating the mass-density over the volume of the sphere

$$M(r) = \underbrace{4\pi}_{} \int_0^R r^2 \, dr \; \rho(r)$$

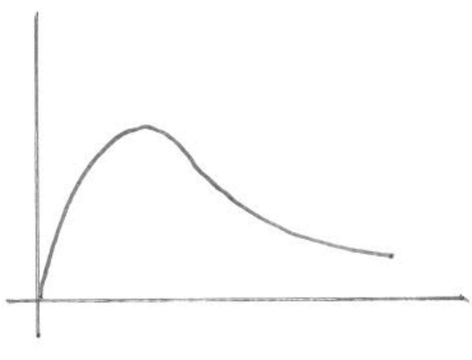
↑ integral over angles.

The resulting velocity profiles are obtained via mathematica (see notebook)

Navaro - Frenk - White:



c) Einasto profile



Exercise 3: Lyman α - lines

The energy of a photon emitted in a transition of an electron from one atomic shell to another is

$$E = R \left(\frac{1}{n_a^2} - \frac{1}{n_1^2} \right)$$

where R is the Rydberg - constant:

$$R = \left(\frac{e^2}{4\pi\varepsilon_0} \right)^2 \frac{m_R}{2\hbar^2}$$

m_R is the reduced mass of the electron - nucleus system:

$$m_R = \frac{m_e m_n}{(m_e + m_n)}$$

(m_n is the mass of the nucleus)

using

$$E = 2\pi\hbar \frac{c}{\lambda}$$

we conclude

$$\lambda_H = \text{const.} \frac{1}{R_H}$$

$$\lambda_D = \text{const.} \frac{1}{R_D}$$

Since the transition is the same, the two proportionality constants are equal and we conclude

$$\frac{\lambda_D}{\lambda_H} = \frac{R_H}{R_D}$$

$$= \frac{m_e m_H}{(m_e + m_H)} \cdot \frac{(m_e + m_D)}{m_e m_D}$$

$$= \frac{m_H}{m_D} \cdot \frac{(m_e + m_D)}{(m_e + m_H)}$$

$$\simeq 0.999\ 728$$

Where the masses given in the problem set are substituted in the last step.

Evaluate

$$\lambda_D = 1215.34\ \text{Å}$$

Then

$$v = c \cdot \frac{\Delta\lambda}{\lambda}$$

$$= c\ \frac{(\lambda_H - \lambda_D)}{\lambda_H} = 81{,}55\ \text{km s}^{-1}$$

agrees with the literature

□