Exercise 2: particles at thermal equilibrium

Fermi - Dirac (+) and Bose - Einstein distribution (-)

$$f = \frac{1}{e^{(E-\mu)/T} \pm 1}$$

a) relativistic limit $\mu \ll T$, $m \ll T$

- integrals are dominated by momentum and we may set $E = |p|$.
- in this limit:

$$f = \frac{1}{e^{p/T} \pm 1}$$

- number density:

$$n = g \cdot \int \frac{d^3 p}{(2\pi)^3} \; \frac{1}{e^{p/T} \pm 1}$$

$$= g \cdot \int_0^\infty \frac{p^2 dp \cdot 4\pi}{(2\pi)^3} \; \frac{1}{e^{p/T} \pm 1}$$

introduce dimensionless integration variable $x = p/T$

$$= \frac{g}{2\pi^2} \, T^3 \int_0^\infty dx \; x^2 \; \frac{1}{e^x \pm 1} \qquad (1)$$

The dimensionless integral evaluates to

$$\int_0^\infty dx \quad \frac{x^2}{e^x + 1} = \frac{3}{2} \zeta(3)$$

$$\int_0^\infty dx \quad \frac{x^2}{e^x - 1} = 2 \zeta(3)$$

Substitution gives:

$$n_F = \frac{3}{4\pi^2} \zeta(3) \, T^3$$

$$n_B = \frac{g}{\pi^2} \zeta(3) \, T^3$$

Energy density:

The analogue of eq. (1) is

$$\rho = g \int \frac{d^3p}{(2\pi)^3} \, |p| \, \frac{1}{e^{p/T} \pm 1}$$

$$= \frac{g}{2\pi^2} T^4 \int_0^\infty dx \, x^3 \, \frac{1}{e^x \pm 1}$$

Evaluate the integrals

$$\int_0^\infty dx \, x^3 \, \frac{1}{e^x + 1} = \frac{7\pi^4}{120}$$

$$\int_0^\infty dx \, x^3 \, \frac{1}{e^x - 1} = \frac{\pi^4}{15}$$

Substitute:

$$\rho_F = \frac{7}{8}\,\frac{\pi^2}{30}\, g\, T^4$$

$$\rho_B = \frac{\pi^2}{30}\, g\, T^4$$

Pressure:

since $|E| = |p|$ the relation $p_i = \rho_i / 3$ is obvious.

The numerical factors are obtained by substitution:

$$p_F = \frac{1}{3}\rho_F$$

$$= \frac{7}{24}\cdot\frac{\pi^2}{30}\cdot\frac{\zeta(3)}{\zeta(3)}\, g\, T^3\, T \cdot \frac{3}{4\pi^2}\cdot\frac{4\pi^2}{3}$$

$$= n_F\, T \cdot \underbrace{\frac{1}{\zeta(3)}\ \frac{7}{6}\frac{\pi^4}{90}}_{=\ 1.0505}$$

analogous:

$$p_B = \frac{1}{3}\rho_B$$

$$= \frac{\pi^2}{90}\cdot\frac{1}{\zeta(3)}\left(\frac{g}{\pi^2}\,\zeta(3)\, T^3\right)\cdot \pi^2\, T$$

$$\simeq 0.9004\, n_B\, T$$

$\square$

b) The non-relativistic limit

· since $e^{(E-\mu)/T} \pm 1 \approx e^{(E-\mu)/T}$

bosons and fermions give the same result in this limit

number density

$$n = g \int \frac{d^3p}{(2\pi)^3} \; e^{\mu/T} \; e^{-(m + \vec{p}^{\,2}/2m)/T}$$

$$= \frac{g}{(2\pi)^3} \cdot 4\pi \; e^{\mu/T} e^{-m/T} \int_0^\infty dp \; p^2 \; e^{-\frac{p^2}{2mT}}$$

dimensionless variable : $x = \frac{p}{\sqrt{2mT}}$

$$= \frac{g}{2\pi^2} \; e^{\mu/T} e^{-m/T} \; (2mT)^{3/2} \int_0^\infty dx \; x^2 \; e^{-x^2}$$

evaluate integral :

$$\int_0^\infty dx \; x^2 \; e^{-x^2} = \frac{\sqrt{\pi}}{4} = I_1$$

substitution yields :

$$n = g \left(\frac{mT}{2\pi} \right)^{3/2} e^{-(m-\mu)/T}$$

□

energy density :

$$\rho = g \cdot \frac{4\pi}{(2\pi)^3} \int p^2 dp \left(m + \frac{p^2}{2m} \right) e^{-\frac{(m-\mu)}{T}} e^{-\frac{p^2}{2mT}}$$

The term containing m is proportional to the n - integral

The new integral is

$$\int_0^\infty dp \; p^4 \; e^{-\frac{p^2}{2mT}}$$

$$= (2mT)^{5/2} \int_0^\infty dx \; x^4 \; e^{-x^2}$$

$$= (2mT)^{5/2} \cdot \frac{3}{2} \cdot \frac{\sqrt{\pi}}{4} = \frac{3}{2} (2mT) \cdot \underbrace{(2mT)^{3/2} I_1}_{\text{Term appearing in } n.}$$

Thus

$$\rho = m \cdot n + \frac{3}{2} \cdot 2Tm \cdot \frac{1}{2m} \cdot n$$

$$= n \left(m + \frac{3}{2} T \right)$$

Pressure :

$$p = g \cdot \frac{4\pi}{(2\pi)^3} \int p^2 dp \cdot \frac{p^2}{3m} e^{-\frac{p^2}{2mT}} \cdot e^{-(m-\mu)/T}$$

$$= \frac{2}{3} \cdot (\text{ second integral appearing in } \rho \text{ })$$

$$= n T$$

$\square$

c) The Friedmann equation for a spatially flat universe is

$$H^2 = \frac{8\pi G}{3} \rho$$

where ρ is the total energy density

· in the radiation dominated era ρ is dominated by relativistic particles. (by definition of radiation dominance)

substituting the results from a):

$$\rho_F = \frac{7}{8} \rho_B = \frac{\pi^2}{30} g T^4$$

g counts relativistic d.o.f. Thus:

$$H^2 = \frac{8\pi G}{3} \cdot \frac{\pi^2}{30} \left(g_{*,b} + \frac{7}{8} g_{*,f} \right) T^4$$

d) At $T = 1\,\mathrm{MeV}$ the relativistic d.o.f. are:

bosons: photons (2 polarizations, $g = 2$)

fermions: electron $g = 2$

positron $g = 2$

3 neutrinos $g = 2$

gives

$$\underset{\gamma}{g_*} = 2 + \underset{e^\pm}{\frac{7}{8} \cdot 2 \cdot 2} + \underset{\nu}{\frac{7}{8} \cdot 3 \cdot 2} = 10.75$$

Exercise 2: Equation of state parameter for photons

a) use that

$$g_{\mu\nu} u^{\mu} u^{\nu} = -1$$

$$g_{\mu\nu} g^{\mu\nu} = 4$$

To obtain

$$T = g_{\mu\nu} T^{\mu\nu}$$

$$= -(\rho + p) + 4p$$

$$= -\rho + 3p$$

b) making all metric tensors explicit, the action (2) reads:

$$S^{\gamma} = -\frac{1}{4} \int d^4x \sqrt{-g}\; F_{\mu\rho} F_{\nu\sigma}\, g^{\mu\nu} g^{\rho\sigma}$$

def. of energy momentum tensor:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S^{\text{matter}}}{\delta g^{\mu\nu}}$$

Perform variation using the identities from class

$$\frac{\delta S^{\gamma}}{\delta g^{\mu\nu}} = -\frac{1}{4} \int_x \Big\{ -\frac{1}{2} \sqrt{-g}\; g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \delta^4(x-y)$$

$$+ 2\sqrt{-g}\; F_{\mu\rho} F_{\nu\sigma}\, g^{\rho\sigma} \delta^4(x-y)$$

Carrying out the spacetime - integrals via the δ- distribution and substituting into the definition of $T_{\mu\nu}$ yields :

$$T_{\mu\nu} = - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + F_{\mu\lambda} F_{\nu}{}^{\lambda}$$

This agrees with Weinberg, Appendix B

□

c) Compute

$$T = g^{\mu\nu} T_{\mu\nu}$$

$$= - \frac{1}{4} \underbrace{\left(g^{\mu\nu} g_{\mu\nu} \right)}_{= 4} F_{\rho\sigma} F^{\rho\sigma} + g^{\mu\nu} F_{\mu\lambda} F_{\nu}{}^{\lambda}$$

$$= 0$$

Comparison with stress - energy tensor of a general perfect fluid (part a)) yields

Photons satisfy the equation of state

$$- \rho + 3 p = 0$$

comparison : $p = w \rho$

yields equation of state parameter $w = 1/3$

□

Exercise 3: Potential slow-roll parameters:

Derive : $\epsilon \simeq \epsilon_V$:

Step 1: The time-derivative of eq. 4b yields

$$2 H \dot{H} = \frac{1}{3 M^2} \left(\frac{1}{2} \cdot 2 \dot{\phi} \ddot{\phi} + V_{,\phi} \dot{\phi} \right)$$

use eq 4a to eliminate $\ddot{\phi}$:

$$2 H \dot{H} = \frac{1}{3 M^2} \dot{\phi} \left(- 3 H \dot{\phi} - V_{,\phi} + V_{,\phi} \right)$$

Thus

$$2 \dot{H} = - \frac{\dot{\phi}^2}{M^2} \qquad (1)$$

The eqs 4 in the slow-roll regime reduce to:

4a:
$$\dot{\phi} \simeq - \frac{1}{3H} V_{,\phi} \qquad (2)$$

4b:
$$H^2 \simeq \frac{1}{3 M^2} V \qquad (3)$$

where $\simeq$ denotes the use of the slow-roll conditions

Then :

$$\varepsilon \equiv - \frac{\dot{H}}{H^2}$$

$$\overset{(1)}{=} \frac{1}{2} \frac{\dot{\phi}^2}{H^2 M^2}$$

$$\overset{(2)}{=} \frac{1}{2} \frac{1}{H^2 M^2} \cdot \frac{(V_{,\phi})^2}{9 H^2}$$

$$= \frac{M^2}{2} \frac{1}{(3 H^2 M^2)^2} (V_{,\phi})^2$$

$$\overset{(3)}{=} \frac{M^2}{2} \left(\frac{V_{,\phi}}{V} \right)^2$$

□

Now reexpress $\eta = - \frac{\ddot{\phi}}{H \dot{\phi}}$

Differentiation of eq. 2 on page - S 6 - :

$$\ddot{\phi} \simeq \frac{1}{3 H^2} \dot{H} \, V_{,\phi} - \frac{1}{3 H} V_{,\phi\phi} \dot{\phi}$$

$$= - \frac{1}{3} \varepsilon V_{,\phi} - \frac{1}{3 H} V_{,\phi\phi} \dot{\phi}$$

substitute into η :

$$\eta \simeq - \frac{1}{H \dot{\phi}} \cdot \left(- \frac{1}{3} \right) \left(\varepsilon V_{,\phi} + \frac{1}{H} V_{,\phi\phi} \dot{\phi} \right)$$

$$\simeq \frac{1}{3} \frac{V_{,\phi\phi}}{H^2} + \frac{1}{3 H \dot{\phi}} V_{,\phi} \varepsilon_V$$

finally use eq (3) on first term and eq (2) on second term to obtain

$$\eta = \underbrace{M^2 \frac{V_{,\phi\phi}}{V}}_{\equiv \eta_V} - \epsilon_V$$

□

Exercise 4 : Single - field inflation

We have :

$$V = \frac{1}{2} m^2 \phi^2$$

a) Slow - roll parameters :

- substitute potential into eq. 5 to obtain

$$\varepsilon_V = 2 \frac{M_{Pl}^2}{\phi^2} \qquad \frac{V'}{V} = \frac{2}{\phi}$$

$$\eta_V = 2 \frac{M_{Pl}^2}{\phi^2} \qquad \frac{V''}{V} = \frac{2}{\phi^2}$$

b) Since $\eta_V = \varepsilon_V$ it suffices to analyze one condition

Setting

$$\varepsilon_V = 1 = 2 \frac{M_{Pl}^2}{\phi_{end}^2}$$

one obtains

$$\phi_{end} = \sqrt{2} M_{Pl}$$

and inflation occurs if $\phi > \phi_{end}$, i.e. the scalar field needs to take transplanckian values

$\Rightarrow$ large - field inflation

c) From the lecture, we have

$$N(\phi) = \int_{\phi_{end}}^{\phi_{init}} \frac{d\phi}{\sqrt{2\epsilon_V}} \cdot \frac{1}{M_{Pl}}$$

$$= \frac{1}{M} \int_{\phi_e}^{\phi_i} \frac{1}{2} \frac{1}{M} \phi \, d\phi$$

$$= \frac{\phi_{init}^2}{4 M_{Pl}^2} - \frac{1}{2}$$

where we substituted $\phi_{end} = \sqrt{2} M_{Pl}$ in the last step.

d) Evaluate:

$$\frac{1}{M_{Pl}} \int_{\phi_{end}}^{\phi_{cmb}} \frac{d\phi}{\sqrt{2\epsilon_V}} = N_{CMB}$$

gives:

$$N_{CMB} = \frac{\phi_{CMB}^2}{4 M_{Pl}^2} \underbrace{- \frac{1}{2}}_{\text{neglect}}$$

$$\phi_{CMB} = 2 M_{Pl} \sqrt{N_{CMB}}$$

$$\simeq 15 \, M_{Pl}$$

Exercise 5: Reheating

The energy density and pressure of a scalar field is given by:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \qquad (*)$$

For a harmonic oscillator whose motion is averaged over one period the averaged kinetic energy equals the averaged potential energy:

$$\frac{1}{2}\langle\dot{\phi}^2\rangle_t = \langle V(\phi)\rangle_t$$

applying this averaging to $(*)$ yields:

$$\langle \overline{\rho}_\phi \rangle_t = \frac{1}{2}\langle\dot{\phi}^2\rangle_t + \langle V(\phi)\rangle_t$$

$$= \langle \dot{\phi} \rangle_t$$

$$\langle p_\phi \rangle_t = \frac{1}{2}\langle\dot{\phi}^2\rangle_t - \langle V(\phi)\rangle_t$$

$$= 0$$

Thus the system behaves like pressureless dust, with equation of state parameter $w = 0$.

From the solution of the continuity equation one then has that the averaged energy density obeys:

$$\bar{\rho}_\phi(t) = \bar{\rho}_\phi(t_I) \cdot \left(\frac{a(t_I)}{a(t)} \right)^3$$

where t_I is the time-scale where the system settled into oscillations around the minimum

Taking the t - derivative

$$\frac{d}{dt} \bar{\rho}_\phi = \bar{\rho}_\phi(t_I) \cdot a(t_I)^3 \cdot (-3) \frac{1}{a(t)^4} \dot{a}$$

$$= -3 H \bar{\rho}_\phi$$

Thus

$$\frac{d}{dt} \bar{\rho}_\phi + 3 H \bar{\rho}_\phi = 0$$

note : This equation may be modified including other decay channels: (into standard model particles)

$$\frac{d}{dt} \bar{\rho}_\phi + 3 (H + \Gamma_\phi) \bar{\rho}_\phi = 0$$