Cosmology 2015: Solutions week 2

Exercise 1: True or false:

1) $$g_{\alpha\beta} = g_{\beta\alpha}$$

true: the spacetime metric is symmetric

2) $$g_{\alpha\beta} = g_{\beta\mu}$$

wrong: the free indices on both sides do not agree

3) $$g_{\alpha\beta}\, dx^\alpha dx^\beta = g_{\alpha\gamma}\, dx^\alpha dx^\beta$$

wrong: same as 2)

4) $$\Gamma^\alpha{}_{\alpha\beta} A^\beta = g_{\alpha\gamma} A^\alpha B^\gamma$$

true: index structure is correct.

5) $$\Gamma^\sigma{}_{\alpha\beta} B^\alpha C^\beta = \Gamma^\sigma{}_{\alpha\beta} B^\beta C^\alpha$$

true: $\Gamma^\sigma{}_{\alpha\beta}$ is symmetric in lower indices

6) $$\Gamma^\alpha{}_{\beta\gamma} A^\alpha C^\beta C^\gamma = B^\alpha$$

wrong: 2 mistakes:

- free indices on lhs / rhs do not match
- one cannot contract two upper indices (there are two upper α indices on rhs)

7) $$\frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta$$

correct.

8) $$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = 0$$

correct equation; note that the equation is not a tensor equation however. It does not transform multiplicative under a change of coordinates.

9) $$\bar{g}_{\alpha\beta}\,\bar{A}^\alpha\,\bar{B}^\beta = g_{\alpha\beta}\,A^\alpha\,B^\beta$$

correct: The quantity transforms as scalar under general coordinate transformations

10) $$\Gamma^\alpha{}_{\alpha\beta} - \Gamma^\beta{}_{\beta\beta} = 0$$

incorrect: again 2 mistakes:

- structure of free indices does not match
- β appears 3 times in $\Gamma^\beta{}_{\beta\beta}$

  $\Rightarrow$ any index can appear only twice (maximum!)

<u>Exercise 2 : illegal index manipulations :</u>

$$g_{\mu\nu} R \stackrel{?}{=} g_{\mu\nu} \left(g^{\mu\nu} R_{\mu\nu} \right)$$

- is wrong : a summation index ($\mu\nu$) is labelled identically to free indices of the equation.

correctly :

$$g_{\mu\nu} R = g_{\mu\nu} \left(g^{\alpha\beta} R_{\alpha\beta} \right) \qquad !$$

line 2 :

$$= \left(g_{\mu\nu} g^{\mu\nu} \right) R_{\mu\nu}$$

Tensor multiplication is associative. What is illegal here is that suddenly the free indices become contracted.

line 3 :

$$g_{\mu\nu} g^{\mu\nu} = 1$$

Is wrong :

$$g_{\mu\nu} g^{\mu\nu} = \delta_{\mu}{}^{\mu} = d$$

The dimension, index range of μ !

Exercise 3: affine connection

- This exercise derives some identities quoted in the lectures.

Note:

$$\bar{g}_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu}$$

and

$$\bar{g}^{\mu\nu} = g^{\alpha\beta} \frac{\partial \bar{x}^\mu}{\partial x^\alpha} \frac{\partial \bar{x}^\nu}{\partial x^\beta}$$

Follows from eq. (14) exchanging barred and unbarred coordinates as well as the lectures.

a)

$$\bar{\Gamma}^{\tau}{}_{\sigma\rho} = \frac{1}{2} \bar{g}^{\tau\lambda} \left[\frac{\partial \bar{g}_{\sigma\lambda}}{\partial \bar{x}^\rho} + \frac{\partial \bar{g}_{\rho\lambda}}{\partial \bar{x}^\sigma} - \frac{\partial \bar{g}_{\sigma\rho}}{\partial \bar{x}^\lambda} \right]$$

substitute transformations:

$$= \frac{1}{2} g^{\alpha\beta} \frac{\partial \bar{x}^\tau}{\partial x^\alpha} \frac{\partial \bar{x}^\lambda}{\partial x^\beta}$$

$$\left[\frac{\partial}{\partial \bar{x}^\rho} \left(g_{\mu\nu} \frac{\partial x^\mu}{\partial \bar{x}^\sigma} \frac{\partial x^\nu}{\partial \bar{x}^\lambda} \right) \right.$$

$$+ \frac{\partial}{\partial \bar{x}^\sigma} \left(g_{\mu\nu} \frac{\partial x^\mu}{\partial \bar{x}^\rho} \frac{\partial x^\nu}{\partial \bar{x}^\lambda} \right)$$

$$\left. - \frac{\partial}{\partial \bar{x}^\lambda} \left(g_{\mu\nu} \frac{\partial x^\mu}{\partial \bar{x}^\rho} \frac{\partial x^\nu}{\partial \bar{x}^\sigma} \right) \right]$$

use chain rule

denote a derivative w.r.t. x^μ by $,\mu$ etc

$$= \frac{1}{2} g^{\alpha\beta} \frac{\partial \bar{x}^\tau}{\partial x^\alpha} \frac{\partial \bar{x}^\lambda}{\partial x^\beta}$$

$$\Big\{ g_{\mu\nu,\gamma} \frac{\partial x^\gamma}{\partial \bar{x}^\rho} \frac{\partial x^\mu}{\partial \bar{x}^\sigma} \frac{\partial x^\nu}{\partial \bar{x}^\lambda}$$

$$+ g_{\mu\nu,\gamma} \frac{\partial x^\gamma}{\partial \bar{x}^\sigma} \frac{\partial x^\mu}{\partial \bar{x}^\rho} \frac{\partial x^\nu}{\partial \bar{x}^\lambda}$$

$$- g_{\mu\nu,\gamma} \frac{\partial x^\gamma}{\partial \bar{x}^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^\rho} \frac{\partial x^\nu}{\partial \bar{x}^\sigma} \Big\}$$

$$+ \frac{1}{2} g^{\alpha\beta} \frac{\partial \bar{x}^\tau}{\partial x^\alpha} \frac{\partial \bar{x}^\lambda}{\partial x^\beta} g_{\mu\nu}$$

$$\Big\{ \frac{\partial^2 x^\mu}{\partial \bar{x}^\rho \, \partial \bar{x}^\sigma} \frac{\partial x^\nu}{\partial \bar{x}^\lambda} + \frac{\partial x^\mu}{\partial \bar{x}^\sigma} \frac{\partial^2 x^\nu}{\partial \bar{x}^\rho \, \partial \bar{x}^\lambda}$$

$$+ \frac{\partial^2 x^\mu}{\partial \bar{x}^\rho \, \partial \bar{x}^\sigma} \frac{\partial x^\nu}{\partial \bar{x}^\lambda} + \frac{\partial x^\mu}{\partial \bar{x}^\rho} \frac{\partial^2 x^\nu}{\partial \bar{x}^\sigma \, \partial \bar{x}^\lambda}$$

$$- \frac{\partial^2 x^\mu}{\partial \bar{x}^\rho \, \partial \bar{x}^\lambda} \frac{\partial x^\nu}{\partial \bar{x}^\sigma} - \frac{\partial^2 x^\nu}{\partial \bar{x}^\lambda \, \partial \bar{x}^\sigma} \frac{\partial x^\mu}{\partial \bar{x}^\rho} \Big\}$$

uses symmetry in $g_{\mu\nu}$

$$= \Gamma^\lambda{}_{\mu\nu} \frac{\partial \bar{x}^\tau}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^\sigma} \frac{\partial x^\nu}{\partial \bar{x}^\rho}$$

$$+ \frac{\partial \bar{x}^\tau}{\partial x^\mu} \frac{\partial^2 x^\mu}{\partial \bar{x}^\rho \, \partial \bar{x}^\sigma}$$

The last term is rewritten as follows:

$$\frac{\partial \bar{x}^{\tau}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial \bar{x}^{\rho} \partial \bar{x}^{\sigma}}$$

$$= \frac{\partial \bar{x}^{\tau}}{\partial x^{\mu}} \left(\frac{\partial}{\partial \bar{x}^{\rho}} \frac{\partial x^{\mu}}{\partial \bar{x}^{\sigma}} \right)$$

$$= \frac{\partial}{\partial \bar{x}^{\rho}} \underbrace{\left(\frac{\partial \bar{x}^{\tau}}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial \bar{x}^{\sigma}} \right)}_{\delta^{\tau}_{\sigma}} - \frac{\partial x^{\mu}}{\partial \bar{x}^{\sigma}} \frac{\partial \bar{x}^{\tau}}{\partial x^{\mu} \partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\rho}}$$

$$= - \frac{\partial \bar{x}^{\tau}}{\partial x^{\mu} \partial x^{\nu}} \frac{\partial x^{\mu}}{\partial \bar{x}^{\sigma}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\rho}}$$

Substitution then proofs eq. (16)

$\square$

b) This is a straightforward substitution exercise:

$$D_{\alpha} g_{\mu\nu}$$

$$= g_{\mu\nu,\alpha} - \Gamma^{\lambda}{}_{\mu\alpha} g_{\lambda\nu} - \Gamma^{\lambda}{}_{\nu\alpha} g_{\lambda\mu}$$

$$= g_{\mu\nu,\alpha}$$

$$- \frac{1}{2} g_{\lambda\nu} g^{\lambda\sigma} (g_{\mu\sigma,\alpha} + g_{\alpha\sigma,\mu} - g_{\mu\alpha,\sigma})$$

$$- \frac{1}{2} g_{\lambda\mu} g^{\lambda\sigma} (g_{\nu\sigma,\alpha} + g_{\alpha\sigma,\nu} - g_{\nu\alpha,\sigma})$$

$$
\begin{aligned}
= \; & g_{\mu\nu,\alpha} \\
& - \frac{1}{2} \left(g_{\mu\nu,\alpha} + g_{\alpha\nu,\mu} - g_{\mu\alpha,\nu} \right) \\
& - \frac{1}{2} \left(g_{\nu\mu,\alpha} + g_{\alpha\mu,\nu} - g_{\nu\alpha,\mu} \right) \\
= \; & 0
\end{aligned}
$$

$\square$

Exercise 4: Computing physical distances

physical distance between $r = 3M$ and $r = 2M$ in the Schwarzschild spacetime

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

Step 1:

- parametrize the radial line with line parameter u:

$$x^\alpha(u) = \begin{pmatrix} 0 \\ uM \\ 0 \\ 0 \end{pmatrix} \quad , \quad u \in [2, 3]$$

Step 2:

- Tangent vector along line

$$\frac{dx^\alpha}{du} = \begin{pmatrix} 0 \\ M \\ 0 \\ 0 \end{pmatrix}$$

Step 3:

- evaluate proper length integral

$$L = \int dL = \int \sqrt{g_{\alpha\beta}(x(u))\, dx^\alpha\, dx^\beta}$$

$$L = \int_2^3 du \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{du} \frac{dx^\beta}{du}}$$

$$= M \int_2^3 du \left(1 - \frac{2M}{Mu}\right)^{-1/2}$$

cont. :

$$L = m \left\{ \sqrt{u}\sqrt{u-2} + 2\log\left(\sqrt{u} + \sqrt{u-2}\right) \right\} \Big|_2^3$$

$$= m \left\{ \sqrt{3} + 2\log\left(\sqrt{3}+1\right) - \log 2 \right\}$$

$$\simeq 3.04\, m$$