Solutions : Problem Set Week # 9

Exercise 1 : Derivation of Friedmann's equations

This is a straightforward, though tedious computation.

a) The non-zero Christoffel symbols for the FRW-metric are:

$$\Gamma^t{}_{rr} = \frac{a\dot{a}}{1-kr^2} \qquad \Gamma^r{}_{tr} = \frac{\dot{a}}{a}$$

$$\Gamma^t{}_{\theta\theta} = r^2 a\dot{a} \qquad \Gamma^r{}_{rr} = \frac{rk}{1-kr^2}$$

$$\Gamma^t{}_{\phi\phi} = r^2 \sin^2\theta \, a\dot{a}$$

$$\Gamma^r{}_{\theta\theta} = -r(1-kr^2)$$

$$\Gamma^r{}_{\phi\phi} = -r\sin^2\theta(1-kr^2)$$

$$\Gamma^\theta{}_{t\theta} = \frac{\dot{a}}{a} \qquad \Gamma^\phi{}_{t\phi} = \frac{\dot{a}}{a}$$

$$\Gamma^\theta{}_{r\theta} = \frac{1}{r} \qquad \Gamma^\phi{}_{r\phi} = \frac{1}{r}$$

$$\Gamma^\theta{}_{\phi\phi} = -\cos\theta\sin\theta \qquad \Gamma^\phi{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$$

This can be quickly verified via mathematica

Evaluate 00 - component:

$$-3\,\frac{\ddot{a}}{a} - \frac{1}{2}(-1)\cdot\frac{6}{a^2}\left(k + \dot{a}^2 + a\ddot{a}\right) = 8\pi G\,\rho$$

$$3\,\frac{\dot{a}^2}{a^2} = 8\pi G\,\rho - 3\,\frac{k}{a^2}$$

$$H^2 = \frac{8\pi G}{3}\,\rho - \frac{k}{a^2} \qquad ✓\ (1)$$

This is the first equation.

Evaluate rr - component:

$$\frac{1}{1-kr^2}\left(2k + 2\dot{a}^2 + a\ddot{a}\right)$$

$$-\frac{1}{2}\,a^2\,\frac{1}{1-kr^2}\,\frac{6}{a^2}\left(k + \dot{a}^2 + a\ddot{a}\right)$$

$$= 8\pi G\cdot a^2\,\frac{1}{1-kr^2}\,p$$

$$\left(2k + 2\dot{a}^2 + a\ddot{a}\right) - 3\left(k + \dot{a}^2 + a\ddot{a}\right) = 8\pi G\,a^2\,p$$

$$-2\,\frac{\ddot{a}}{a} - \underbrace{\left(H^2 + \frac{k}{a^2}\right)}_{\frac{8\pi G}{3}\rho \text{ from eq (1)}} = 8\pi G\,p$$

gives
$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G\left(\rho + 3p\right) \qquad ✓$$

The resulting Ricci - tensor is:

$$R_{00} = -3\,\frac{\ddot{a}}{a}$$

$$R_{rr} = \frac{1}{1-kr^2}\left(2k + 2\dot{a}^2 + a\ddot{a}\right)$$

$$R_{\theta\theta} = r^2\left(2k + 2\dot{a}^2 + a\ddot{a}\right)$$

$$R_{\phi\phi} = r^2 \sin^2\theta \left(2k + 2\dot{a}^2 + a\ddot{a}\right)$$

Thus:

$$R = \frac{6}{a^2}\left(k + \dot{a}^2 + a\ddot{a}\right)$$

Lower index on $T^{\mu}{}_{\nu}$:

$$T_{\mu\nu} = g_{\mu\lambda}\, T^{\lambda}{}_{\nu}$$

$$= \mathrm{diag}\left(\rho,\ \frac{a^2}{1-kr^2}\, p,\ a^2 r^2 p,\ a^2 r^2 \sin^2 p\right)$$

Now plug into Einsteins equations:

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}$$

b) we analyze the zero - component of $D_\mu T^{\mu 0} = 0$:

$$D_\mu T^\mu{}_\nu =$$

$$\partial_\mu T^\mu{}_\nu + \Gamma^\mu{}_{\lambda\mu} T^\lambda{}_\nu - \Gamma^\lambda{}_{\nu\mu} T^\mu{}_\lambda = 0$$

· used that covariant derivative is metric compatible

specialize to $\nu = 0$

$$\underbrace{\partial_\mu T^\mu{}_0} + \Gamma^\mu{}_{0\mu} \rho - \Gamma^\lambda{}_{0\mu} T^\mu{}_\lambda = 0$$

$$-\dot{\rho} - \rho \cdot 3 \frac{\dot{a}}{a} - p \cdot 3 \frac{\dot{a}}{a} = 0 \qquad (2)$$

gives upon multiplying with a^3

$$\frac{d}{dt} (\rho a^3) = - p \frac{d}{dt} a^3$$

$\square$

c) multiply eq. (1) by a^2:

$$\dot{a}^2 = \frac{8\pi G}{3} \rho a^2 - k$$

Take ∂_t- derivative:

$$2 \dot{a} \ddot{a} = \frac{8\pi G}{3} a^2 \dot{\rho} + \frac{8\pi G}{3} \rho \, 2 a \dot{a}$$

Eliminate $\dot{\rho}$ via eq. (2) conservation law:

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3} a^2 \left(-3\rho\frac{\dot{a}}{a} - 3p\frac{\dot{a}}{a} + 2\rho\frac{\dot{a}}{a}\right)$$

gives upon dividing by $2\dot{a}a$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

which is the second eq. (5).

Exercise 2: Big bang singularity

- equation of state: $p(t) = w\,\rho(t)$

  substitute into (4):

$$\frac{d}{dt}\left[\rho\, a^3\right] = -w\,\rho\,\frac{d}{dt}a^3$$

$$a^3\dot{\rho} + 3\rho a^2\dot{a} + w\rho \cdot 3a^2\dot{a} = 0$$

$$a\dot{\rho} + 3(1+w)\rho\dot{a} = 0$$

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

Integration gives $\rho(t)$ in terms of the scale-factor

$$\rho(t) = \rho_0\left(\frac{a(t_0)}{a(t)}\right)^{3(1+w)}$$

where ρ_0 and $a(t_0) = a_0$ are density and scalefactor at $t_0 \equiv$ today.

- Friedman equation for flat universe $k=0$

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi}{3}\rho(t) = 0$$

$$\dot{a}^2 - \frac{8\pi}{3}a^2\cdot\rho_0\left(\frac{a_0}{a}\right)^{3(1+w)} = 0$$

set

$$c = \frac{8\pi}{3} \, \rho_0 \, a_0^{3(1+w)} > 0$$

gives

$$\frac{da}{dt} = \sqrt{c} \quad a^{1 - \frac{3}{2}(1+w)}$$

NR :

$$1 - \frac{3}{2} - \frac{3}{2} w = - \frac{1}{2} (1 + 3w)$$

$$\frac{da}{a^{-\frac{1}{2}(1+3w)}} = \sqrt{c} \, dt$$

gives :

$$\frac{2}{3} \frac{1}{(1+w)} \, a^{3/2 (1+w)} = \sqrt{c} \, t$$

$$a \sim t^{2/3 \frac{1}{1+w}}$$

For $t = 0$ this goes to zero for $w > -1$

- thus if $w > -1$
  - the scale factor goes to zero
  - $\rho(t) \sim t^{-2}$

    diverges, so we have a Big Bang singularity

b) The de Sitter case corresponds to $w = -1$:

$$\rho = \rho_v = \frac{\Lambda}{8\pi} > 0$$

In this case:

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{3}\Lambda = 0$$

with solution

$$a(t) = a_0 e^{H_0(t - t_0)}$$

$$H_0 = \sqrt{\frac{\Lambda}{3}}$$

For $t \to -\infty$ the scale factor goes to zero, but the energy density remains constant.

$\Rightarrow$ singularity, but no Big Bang.

Exercise 3.: model universe

a) Today at t_0 the Friedman equation reads

$$\underbrace{\left(\frac{\dot{a}}{a}\right)^2 \Bigg|_{t=t_0}}_{H_0^2} - \frac{8\pi}{3} \rho_{total} = 0$$

Solve for ρ_{total}:

$$\rho_{total} = \rho_{crit} = \frac{3H_0^2}{8\pi}$$

b) For the model we have

$$\rho_{total} = \rho_m + \rho_v$$

From exercise 1) we got the scale dependence

$$\rho_m(t) = \rho_{m_0}\left(\frac{a_0}{a}\right)^3 \qquad \rho_v = const$$

Then

$$\rho_{total} = \rho_{crit}\left(\frac{\rho_m(t)}{\rho_{crit}} + \frac{\rho_v}{\rho_{crit}}\right)$$

$$= \rho_{crit}\left(\frac{\rho_{m_0}}{a^3 \rho_{crit}} + \frac{\rho_v}{\rho_{crit}}\right)$$

$$= \rho_{crit}\left(\Omega_m / a^3 + \Omega_v\right)$$

c) Write

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi}{3} \rho_{total} = 0$$

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi}{3} \rho_{crit} \left(\frac{\Omega_m}{a^3} + \Omega_v\right) = 0$$

$$\left(\frac{\dot{a}}{a}\right)^2 - H_0^2 \left(\frac{\Omega_m}{a^3} + \Omega_v\right) = 0$$

$$H_0^{-2} \dot{a}^2 - a^2 \left(\frac{\Omega_m}{a^3} + \Omega_v\right) = 0$$

Thus the equation becomes

$$H_0^{-2} \dot{a}^2 - \mathcal{U}_{eff}(a) = 0$$

where

$$\mathcal{U}_{eff} = \left(a^{-1} \Omega_m + \Omega_v a^2\right)$$

The implicit equation can be found by separation of variables

$$\frac{da}{\sqrt{\mathcal{U}_{eff}}} = H_0 \, dt$$

The integral can be solved giving

$$\frac{2}{3\sqrt{\Omega_v}} \left(\log\left(a^{3/2} \Omega_v + \sqrt{\Omega_v} \sqrt{\Omega_m + a^3 \Omega_v} \right) - \log\left(\Omega_v + \sqrt{\Omega_v} \right) \right) = H_0 (t - t_0)$$

d) We write the $\ddot{a}$ - equation in terms of densities

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi \left(\rho_{total} + 3\, p_{total} \right)$$

- matter is pressureless
- $p_v = -\rho_v$

gives

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi \left(\rho_m + \rho_v - 3\rho_v \right)$$

$$= -\frac{4}{3}\pi\, \rho_{crit} \left(\Omega_m - 2\,\Omega_v \right)$$

This gives the condition:

$$\Omega_v > \frac{1}{2}\,\Omega_m$$

for an universe accelerating today.

Exercise 4: Light from a distant galaxy

- We align our coordinates that the galaxy is located on the x-axis
- From the line element we obtain that the proper distance today is

$$d = a(t_0)\,\Delta x$$

where Δx is the coordinate distance travelled by the light ray: This is obtained from:

$$ds^2 = 0$$
$$= dt^2 - a^2(t)\,dx^2$$

$$\Rightarrow \quad dx = \frac{1}{a(t)}\,dt$$

Integrating gives

$$\Delta x = \int_{x_e}^{x_0} dx = \int_{t_e}^{t_0} \frac{dt}{a(t)}$$

Thus

$$d = a(t_0) \cdot \int_{t_e}^{t_0} \frac{dt}{a(t)}$$

$\square$

Exercise 5 : Vacuum energy from Quantum Gravity

a) we have the units

$$[\hbar] = m^2 \, kg \, s^{-1}$$

$$[G] = m^3 \, kg^{-1} \, s^{-2}$$

$$[c] = m \, s^{-1}$$

Thus

$$[\rho_{Pl}] = m^5 \, s^{-5} \, m^{-2} \, kg^{-1} \, s \, m^{-6}$$
$$kg^2 \, s^4$$
$$= kg \, m^{-3}$$

correct units for a mass density

b) In terms of numbers :

$$\rho_c = 1.88 \cdot 10^{-26} \, h^2 \, \frac{kg}{m^3}$$

$$h = 0.72$$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = 0.7$$

and

$$G = 6.7 \cdot 10^{-11} \, \frac{m^3}{kg \, s^2}$$

$$\hbar = 10^{-34} \, kg \, m^2 \, s^{-1}$$

$$c = 3 \cdot 10^8 \, m \, s^{-1}$$

gives

$$\rho_{Pl} = \frac{c^5}{\hbar G^2}$$

$$\simeq 5.4 \cdot 10^{96} \ \mathrm{kg} \ \mathrm{m}^{-3}$$

$$\rho_\Lambda = \Omega_\Lambda \, \rho_{crit}$$

$$= 0.7 \cdot (0.72)^2 \cdot 1.9 \cdot 10^{-32} \ \mathrm{kg} \ \mathrm{m}^{-3}$$

$$\simeq 6.9 \cdot 10^{-33}$$

Thus

$$\frac{\rho_\Lambda}{\rho_{Pl}} = 1.2 \cdot 10^{-123}$$

This is a huge discrepancy !