Exercise 3 : Reheating the cosmic plasma :

a) 10 GeV is above the confinement / deconfinement phasetransition ; we have free quarks and anti-quarks :

The contribution is then :

fermions :

quarks : (up, down, strange, charm, bottom; Top is too massive to be relativistic)

with 3 colors and 2 spins

$g_* = 5 \cdot 3 \cdot 2 = 30$

antiquarks : $g_* = 30$

same as above

leptons : $e, \nu_e, \mu, \nu_\mu, \tau, \nu_\tau$

with 2 spin states

$g_* = 2 \cdot 6 = 12$

anti - leptons : $g_* = 12$

bosons :

- photons : 2 polarizations $g_* = 2$
- gluons : eight color combinations , two spins

$3 \otimes \bar{3} = 8 \oplus 1$

$g_* = 16$

Thus at $T = 10\,GeV$:

$$g_* = \underset{\gamma}{\underset{\uparrow}{2}} + \underset{\text{gluons}}{\underset{\uparrow}{16}} + \frac{7}{8}\,(\underset{q}{\underset{\uparrow}{30}} + \underset{\bar{q}}{\underset{\uparrow}{30}} + \underset{e}{12} + \underset{\bar{e}}{12})$$

$$= 91.5$$

Today entropy comes from photons and neutrinos

- photons : $g_* = 2$

  neutrinos : $\frac{7}{8} \cdot 3 \cdot 2 \cdot \left(\frac{4}{11}\right)^{4/3} = 1.36$

Thus

$$g_* = 3.36$$

- neutrinos decouple after e^+e^- - annihilation which leads to an effective neutrino temperature

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T$$

yielding the correction factor to g_* neutrino above b)

$$s = \frac{\rho + p}{T}$$

since only relativistic particles contribute ($p = \frac{1}{3} \rho$)

$$s = \frac{4}{3} \frac{\rho}{T}$$

$$= \frac{4}{3} \cdot \frac{\pi^2}{30} g_* (T) \cdot \frac{T^4}{T}$$

$$= \frac{4}{3} \frac{\pi^2}{30} g_* (T) T^3$$

Note: $a^3 s$ is conserved thus

$$\frac{4}{3} \frac{\pi^2}{30} g_* (T) a^3 T^3 \Big|_{10 \text{ GeV}} = \frac{4}{3} \frac{\pi^2}{30} g_* (T) a^3 T^3 \Big|_{\text{today}}$$

Thus

$$\frac{a^3 T^3 \big|_{\text{today}}}{a^3 T^3 \big|_{10 \text{ GeV}}} = \frac{g_* (10 \text{ GeV})}{g_* (\text{today})} = \frac{91,5}{3.36}$$

$$\simeq 27.23$$

Exercises : Cosmology

Solutions : Week 12

Exercise 1: Vacuum fluctuations of the harmonic oscillator

We use

$$\hat{a}|0\rangle = 0 \qquad \leftrightarrow \qquad \langle 0|\hat{a}^+ = 0$$

$$[\hat{a}, \hat{a}^+] = 1$$

Then

$$\langle 0| \hat{x}^2 |0\rangle$$

$$= \langle 0| (v\hat{a} + v^*\hat{a}^+)(v\hat{a} + v^*\hat{a}^+)|0\rangle$$

$$= |v|^2 \langle 0| \hat{a}\hat{a}^+ |0\rangle$$

$$= |v|^2 \langle 0| [\hat{a}, \hat{a}^+] |0\rangle$$

$$= |v|^2 \langle 0|0\rangle$$

$$= |v|^2$$

$\square$

Solutions : Exercise cosmology week 11

Exercise 1: decoupling of Fourier modes

Under translations the perturbations $\delta Q(t, x)$ transform into

$$\delta Q(t, x) \to \delta Q(t, x + \Delta x)$$

Compute the Fourier - transformation of shifted perturbation

$$\delta Q'(t, k) \equiv \int d^3x \quad \delta Q(t, x + \Delta x) e^{-ikx}$$

new integration variable $y = x + \Delta x$

$$= \int d^3y \quad \delta Q(t, y) \quad e^{-ik(x + \Delta x - \Delta x)}$$

$$= e^{ik\Delta x} \int d^3y \; \delta Q(t, y) \, e^{-iky}$$

$$= e^{ik\Delta x} \quad \delta Q(t, k)$$

Intuitively this is clear: shifting the integration variable induces a phase - shift in fourier space.

Writing eq. (3) in the primed coordinate system gives

$$\delta Q_I'(t_2, k) = \sum_N \int d^3\bar{k} \quad e^{-ik\Delta x} \quad T_{IJ}(t_2, t_1, k, \bar{k})$$

$$e^{i\bar{k}\Delta x} \quad \delta Q_J(t, \bar{k})$$

$$\equiv \sum_N \int d^3\bar{k} \quad T'_{IJ}(t_2, t_1, k, \bar{k}) \; \delta Q_J(t, \bar{k})$$

Owed to translation invariance the transfer matrices must be the same in both coordinate systems:

$$T_{IJ}(t_2, t_1, k, \bar{k}) = e^{i\Delta x(\bar{k} - k)} T_{IJ}(t_2, t_1, k, \bar{k})$$

This must hold for all values Δx.

Hence either $\bar{k} = k$ or $T_{IJ}(t_2, t_1, k, \bar{k}) = 0$

Thus the off-diagonal (in $k, \bar{k}$) elements in T_{IJ} must vanish establishing that the Fourier modes decouple.

Exercise 2 : Power spectrum for $m^2 \phi^2$ - inflation

a) From last weeks problem set, we have the slow-roll parameters

$$\epsilon_V = \eta_V = 2 \left(\frac{M_{Pl}}{\phi} \right)^2$$

we are interested in $\phi = \phi_{CMB}$ the field value when the fluctuations of the CMB were created

Evaluating :

$$N_{CMB} = \frac{1}{M_{Pl}} \int_{\phi_{end}}^{\phi_{CMB}} \frac{d\phi}{\sqrt{2\epsilon_V}}$$

$$= \frac{1}{4M^2} \phi^2_{CMB} \underbrace{- \frac{1}{2}}_{\text{neglect}}$$

gives

$$\phi^2_{CMB} = 4 M^2 N_{CMB} \qquad (1)$$

Substituting into the slow-roll parameters then yields

$$\epsilon_V = \eta_V = \frac{1}{2 N_{CMB}}$$

□

b) The dimensionless scalar power spectrum for slow-roll inflation is:

$$\Delta_S^2(k) \simeq \frac{1}{24\pi^2} \frac{V}{M_{Pl}^4} \frac{1}{\varepsilon_V} \Big|_{\text{evaluated at CMB formation}}$$

substitute potential and slow-roll-parameter

$$= \frac{1}{24\pi^2} \frac{1}{M_{Pl}^4} \cdot \left(\frac{1}{2} m^2 \phi_{CMB}^2 \right) \cdot \frac{1}{2} \frac{1}{M_{Pl}^2} \phi_{CMB}^2$$

use eq. (1) to replace ϕ_{CMB}^2 by N_{CMB}

$$= \frac{1}{24\pi^2} \frac{1}{M_{Pl}^6} \cdot \frac{1}{4} \cdot m^2 \cdot 16 \cdot M_{Pl}^4 \, N_{CMB}^2$$

$$= \frac{1}{6\pi^2} \frac{1}{M_{Pl}^2} \cdot m^2 \, N_{CMB}^2$$

Solve for m and substitute $\Delta_S \simeq 10^{-9}$

$$m = \underbrace{\frac{\sqrt{6}\,\pi}{N_{CMB}} \cdot \sqrt{10^{-9}}}_{= 4 \cdot 10^{-6}} \; M_{Pl}$$

gives result quoted in exercise.

c) We have:

$$n_S = 1 + 2\eta_V^* - 6\varepsilon_V^* = 1 - \frac{2}{N_{CMB}} = 0.96$$

$$r = 16\,\varepsilon_V^* = \frac{8}{N_{CMB}} = 0.1$$

d) Current best estimates give

$$n_S = 0.958 \pm 0.008$$

$\Rightarrow$ very good agreement with the predictions from the very simple model!

Exercise 3 : The Mukhanov - action

a) Starting point :

$$S^{(2)} = \frac{1}{2} \int d^4x \; a^3 \; \frac{\dot{\phi}^2}{H^2} \left(\dot{\mathcal{R}}^2 - a^{-2} (\partial_i \mathcal{R})^2 \right)$$

Change to Mukhanov - variables:

$$v = z\,\mathcal{R} \quad , \quad z^2 = a^2 \frac{\dot{\phi}^2}{H^2} \quad , \quad dt = a\, d\tau$$

$$v' = \frac{dv}{d\tau} = \frac{dv}{dt}\frac{dt}{d\tau} = z'\,\mathcal{R} + a z\,\dot{\mathcal{R}}$$

Thus

$$\dot{\mathcal{R}} = \frac{v' - z'\mathcal{R}}{a z}$$

note : z is a function of t (or τ) only thus

$$\partial_i \mathcal{R} = z^{-1} \partial_i v$$

Substitute

$$S^{(2)} = \frac{1}{2} \int d^3x \; (a\, d\tau) \cdot a z^2$$

$$\left\{ \frac{1}{(az)^2} (v' - z'\mathcal{R})^2 - \frac{1}{a^2} \frac{1}{z^2} (\partial_i v)^2 \right\}$$

$$= \frac{1}{2} \int d^3x\, d\tau \left\{ \left(v' - \frac{z'}{z} v \right)^2 - (\partial_i v)^2 \right\}$$

First term in $\{\dots\}$ brackets

$$\int_x \left\{ (v')^2 - 2 \frac{z'}{z} v' v + \left(\frac{z'}{z}\right)^2 v^2 \right\}$$

$$= \int_x \left\{ (v')^2 - 2 \frac{z'}{z} \cdot \frac{1}{2} \frac{d}{d\tau} (v^2) + \left(\frac{z'}{z}\right)^2 v^2 \right\}$$

$$\overset{\text{P.I.}}{=} \int_x \left\{ (v')^2 + \frac{d}{d\tau}\left(\frac{z'}{z}\right) \cdot v^2 + \left(\frac{z'}{z}\right)^2 v^2 \right\}$$

$$= \int_x \left\{ (v')^2 + \frac{z''}{z} v^2 \right\}$$

Thus:

$$S^{(2)} = \frac{1}{2} \int d^3x \, d\tau \left\{ (v')^2 - (\partial_i v)^2 + \frac{z''}{z} v^2 \right\}$$

$\square$

- agrees with (6)
- has a sign mistake compared to (183)

b) In position space the equations of motion are

$$-v'' + \partial_i \partial_i v + \frac{z''}{z} v = 0$$

The conventions for Fourier-transforms follow from (2):

$$v(\tau, k) = \int d^3x \; v(\tau, x) \, e^{-ikx}$$

$$v(\tau, x) = \int \frac{d^3 p}{(2\pi)^3} \; v(\tau, k) \, e^{ikx}$$

Substitute the second relation yields e.o.m for the Fourier - mode :

$$v_k'' + k^2 v_k - \frac{z''}{z} v_k = 0$$

equivalently :

$$v_k'' + \left(k^2 - \frac{z''}{z} \right) v_k = 0$$