Cosmology exercises & Solutions

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Contents

Werkcollege, Cosmology 2016/2017, Week 4

These are the exercises and hand-in assignment for the 4th week of the course *Cosmology*. Every week, one of the problems provides credit towards the final exam. If at least **10** of these problems are handed in and approved, one problem on the final exam may be skipped. The hand-in assignment for this week is **Problem 4.2** below.

Flux and intensity

In the lecture we discussed the fundamental concepts *flux* (the energy passing through a unit surface per unit time) and *intensity* (flux per solid angle). In S.I. units, flux is expressed in units of [$\mathrm{W\,m^{-2}}$] and intensity in [$\mathrm{W\,m^{-2}\,sr^{-1}}$]. In many practical applications, it is more useful to specify the *flux density* (or *specific flux*), i.e., the flux per unit wavelength (F_λ) or frequency (F_ν) interval, and the corresponding quantities for the intensity (I_λ, I_ν). The flux density is often expressed in Jy, where 1 Jy = $10^{-26}\,\mathrm{W\,m^{-2}\,Hz^{-1}}$. The total flux is then, in principle, found by integration over all wavelengths (or frequencies):

$$F = \int F_\lambda \,\mathrm{d}\lambda \tag{4.0.1}$$

or

$$F = \int F_\nu \,\mathrm{d}\nu \tag{4.0.2}$$

In practice, measuring the specific flux at all wavelengths is difficult, and we sometimes also refer to the flux integrated over a specific wavelength region, such as a particular photometric band.

4.1 F_λ and F_ν

Show that F_ν and F_λ are related as

$$\frac{F_\nu}{F_\lambda} = \frac{c}{\nu^2} = \frac{\lambda^2}{c} \tag{4.1.3}$$

4.2 Flux of astronomical objects

The brightest star in the sky, Sirius, has a radius of about 1.75 $R_\odot$ and a temperature of 9900 K. Its distance is 2.6 pc.

1. Approximating the spectrum of Sirius by a black-body with $T_{\mathrm{eff}} = 9900$ K, calculate the specific intensity I_λ of light emitted at 5500 Å (i.e. at the centre of the V-band)

2. Making the approximation that I_λ is constant over the wavelength range covered by the V-band, and assuming that the bandwidth is $\Delta\lambda_V = 900$ Å, what is the V-band intensity I_V of the light emitted by Sirius?

3. Calculate the V-band flux from Sirius measured above the Earth's atmosphere. (*Hint:* you may make use of the fact that the intensity of black-body radiation is independent of the viewing angle. The integral $\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \frac{1}{2}$ might be useful).

4. How many V-band photons would enter the aperture of the Hubble Space Telescope per second if it were pointed at Sirius? Assume that HST has a circular aperture with a diameter of 2.4 m. You can also assume that all photons have the same energy, corresponding to $\lambda = 5500$ Å.

4.3 Flux, Magnitude and Surface Brightness

The flux density received from the star Vega (above the Earth's atmosphere) is $F_\nu = 3.6 \times 10^{-23}$ W m^{-2} Hz^{-1} in the visual region of the spectrum. Vega has a visual magnitude $m_V = 0$.

1. Calculate the visual magnitude of a source with a flux density of $F = 1$ Jy
2. The faintest stars visible to the unaided eye under a dark sky have visual magnitudes $V \approx 6$. Calculate the limiting sensitivity of the eye in Jy.
3. In astronomy, the term *surface brightness* is sometimes used instead of intensity. The natural night sky has an average visual surface brightness of about 22 mag arcsec^{-2} at new Moon (meaning that the flux received from one square arcsecond of blank sky is the same as that received from a 22nd magnitude star). Over how large an area of the sky does one need to integrate to get a flux similar to that of the faintest naked-eye stars?

Formulae and constants

Black-body radiation:

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

$$I_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

Radius of the Sun: $R_\odot = 7 \times 10^8$ m

1 pc = 3.09×10^{16} m

Planck's constant: $h = 6.626 \times 10^{-34}$ m^2 kg s^{-1}

Boltzmann's constant: $k = 1.38 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}

Werkcollege, Cosmology 2016/2017, Week 5

These are the exercises and hand-in assignment for the 5th week of the course *Cosmology*. Every week, one of the problems provides credit towards the final exam. If at least **10** of these problems are handed in and approved, one problem on the final exam may be skipped. The hand-in assignment for this week is **Problem 5.5** below.

5.1 Distance and distance modulus

Show that an error or uncertainty of 0.1 magnitudes in the distance modulus, $m - M$, is roughly equivalent to a 5% error in the distance, D

5.2 Moving cluster method

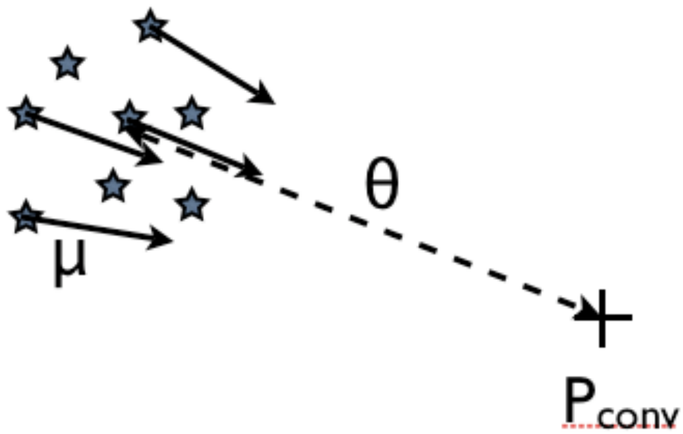

Fig. 1: The star cluster discussed in Problem 5.2.

A star cluster is observed to have a proper motion $\mu = 0.110'' \text{ yr}^{-1}$ and radial velocity $v_r = 40$ km s^{-1}. The proper motions of stars in the cluster appear to be converging towards the point P_{conv}, located at an angle of $\theta = 30°$ from the centre of the cluster on the sky.

1. Calculate the distance to the cluster
2. What was the smallest distance between the Sun and the cluster, relative to the current distance?
3. When did the closest passage occur? Show that this can be calculated without knowing the distance of the cluster!
4. Assuming effects of stellar evolution and extinction are negligible, when will the apparent brightness of the cluster have decreased by 1 magnitude?

5.3 Cepheids

The relation between the mean apparent visual magnitude m_V and the period P (in days) for Cepheids in the Large Magellanic Cloud (LMC) is observed to be

$$m_V = -2.5 \log_{10} P + 17.0 \tag{5.3.1}$$

For the Galactic Cepheid δ Cep, a trigonometric parallax of 3.8×10^{-3} arcseconds is observed. δ Cep has $\log_{10} P = 0.73$ and a mean apparent magnitude $m_V = 3.8$.
In the following, assume that Cepheids everywhere follow a universal period-luminosity relation. You can ignore the effects of interstellar extinction (or, to put it differently, assume that all measurements have been corrected for this effect).

1. Find the distance to the LMC.

2. A Cepheid in the galaxy M100 has apparent mean magnitude $m_V = 27.1$ and period $P = 10$ days. Find the distance to M100.

5.4 Baade-Wesselink method

This exercise is taken from the book "Galactic Dynamics", J. Binney & M. Merrifield

A star expands in a spherically-symmetric manner with radial velocity v_r. Defining a spherical coordinate system on the surface of the star with the polar axis aligned along the line of sight, show that the measurable flux-weighted mean line-of-sight velocity will be

$$v_{\rm los} = v_r \frac{\int_0^{\pi/2} I(\theta) \cos^2\theta \sin\theta \, d\theta}{\int_0^{\pi/2} I(\theta) \cos\theta \sin\theta \, d\theta} \tag{5.4.1}$$

Hence show that, for a star of uniform brightness, $p = v_r/v_{\rm los} = 1.5$. In reality, a star will not appear uniformly bright: its opacity means that near the edge of the star (its "limb") one cannot peer so far into its atmosphere, so one sees the less bright outer layers. A reasonable analytic approximation to this **limb darkening** is given by $I(\theta) = I(0)(0.4 + 0.6 \cos\theta)$. In this approximation, show that $p = 24/17$.

5.5 K-corrections

The K-correction is the difference between the observed magnitude $m_{\rm obs}(z)$ for a source at redshift z and the magnitude that would be observed if the source were at rest, $m_{\rm rest}$:

$$m_{\rm rest} = -2.5 \log_{10} \int f(\lambda) S(\lambda) \, d\lambda + \text{const} \tag{5.5.1}$$

$$m_{\rm obs}(z) = -2.5 \log_{10} \int f(\lambda') S[\lambda'(1+z)] \, d\lambda' + \text{const} \tag{5.5.2}$$

$$\tag{5.5.3}$$

In addition to the redshift z, the K-correction depends on the spectrum of the source (here expressed as a function of wavelength, $f[\lambda]$) and the spectral response of the system used for the observations, $S(\lambda)$. The K-correction is a purely instrumental effect that simply accounts for the fact that light emitted at wavelength λ' is observed at wavelength λ. It does *not* take into account the cosmological effects of the redshift due to the expansion of the Universe.

1. Show that (5.5.2) and (5.5.1) lead to the following expression for the K-correction:

$$K = m_{\mathrm{obs}} - m_{\mathrm{rest}} \tag{5.5.4}$$

$$= 2.5 \log_{10} \frac{\int f(\lambda) S(\lambda)\, \mathrm{d}\lambda}{\int f[\lambda/(1+z)] S(\lambda)\, \mathrm{d}\lambda} + 2.5 \log_{10}(1+z) \tag{5.5.5}$$

2. Find and write down the equivalent expression for the K-correction in terms of the spectrum as a function of *frequency*, $f(\nu)$

3. Calculate the K-correction for a source with a power-law spectrum, $f(\lambda) \propto \lambda^{\beta}$. To simplify the calculations, you can approximate the bandpass transmission curve $S(\lambda)$ as a box function, i.e., assume that $S(\lambda)$ is a (positive) constant for $\lambda_1 < \lambda < \lambda_2$ and zero elsewhere.

Formulae and constants

Distance modulus (D in pc):

$$m - M = 5 \log_{10} D - 5$$

Black-body radiation:

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

$$I_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

Radius of the Sun: $R_\odot = 7 \times 10^8$ m

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Boltzmann's constant: $k = 1.38 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}

Werkcollege, Cosmology 2016/2017, Week 6

These are the exercises and hand-in assignment for the 6th week of the course *Cosmology*. Every week, one of the problems provides credit towards the final exam. If at least **10** of these problems are handed in and approved, one problem on the final exam may be skipped. The hand-in assignment for this week is **Problem 6.1** below.

6.1 Mass distribution of the Milky Way

a. Assuming that the mass distribution in the Milky Way is dominated by a spherically symmetric dark matter halo, show that a flat rotation curve implies the following density profile:

$$\rho_h(R) = \frac{v_c^2}{4\pi G} R^{-2} \qquad (6.1.1)$$

where R is the galactocentric distance and v_c the circular velocity.

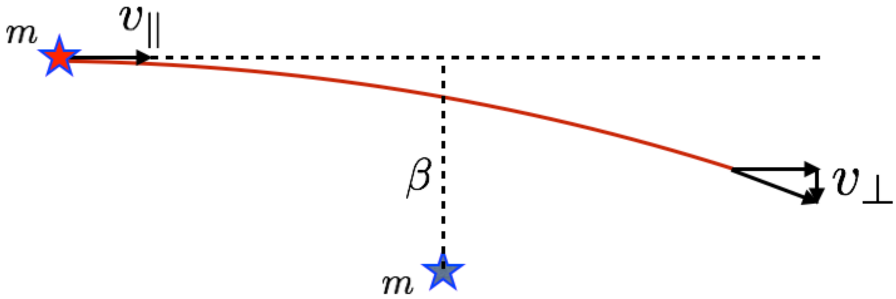
b. For a circular velocity $v_c = 200$ km/s, $R_0 = 8$ kpc, calculate the density of the dark matter halo near the Sun.

In reality, other components of the Milky Way make non-negligible contributions to the mass. Near the Sun, the density of the stellar disc is about $\rho_d(R_0) = 0.08\,M_\odot\,\mathrm{pc}^{-3}$. Assume that the Sun is located at the midplane of the disc and that the vertical density distribution of the disc is exponential with scale height $z_{\rm scl} = 300$ pc.

c. At R_0, how far above the Galactic plane, z, is the density of the disc equal to that of the dark halo estimated above? (you may assume that ρ_h is independent of z for fixed $R = R_0$). If you did not find an answer in **6.1.b** you may assume $\rho_h(R_0) = 0.02\,M_\odot\,\mathrm{pc}^{-3}$ but note that this is *not* the correct answer.

6.2 Two-body relaxation

The process of two-body relaxation plays a very important role in stellar dynamics. Over time, it drives the distribution of stellar velocities towards a Maxwellian equilibrium distribution, so that any memory of the initial conditions will eventually be erased. For the typical stellar densities and relative velocities encountered in galaxies the two-body relaxation time scale is, however, very long, so that present-day galaxies still retain some memory of their formation conditions. In this exercise we go through the derivation of the two-body relaxation time scale.



Consider an encounter between two stars. Assume for simplicity that both stars have the same mass, m. We use a coordinate system in which one star is initially moving along a straight line with velocity $v_{||}$, equal to the typical relative velocities V of stars in the system, and the other is stationary. Continuing along this path, the minimum separation between the two stars ((the *impact parameter*) will be β (see figure), and the star will experience an acceleration a due to the mutual gravitational attraction between the two stars. Clearly, by Newton's 3rd law, the other star will experience an acceleration of the same magnitude but opposite direction. The component of a perpendicular to $v_{||}$, $a_\perp$, will produce a net velocity $v_\perp$ perpendicular to $v_{||}$ after the encounter.

One distinguishes between *strong* and *weak* encounters, where an encounter is said to be *strong* if the smallest distance of the stars during the encounter (β) is such that the (absolute) potential energy $|U(\beta)|$ is equal to (or greater than) the mean kinetic energy of a star.

a. Show that a strong encounter corresponds to an impact parameter

$$\beta < \frac{2Gm}{V^2} \tag{6.2.1}$$

In the solar neighbourhood, the mean volume density of stars is about $n = 0.1\ \mathrm{pc}^{-3}$. Typical relative velocities are 10 km/s, and the average mass of a star can be taken to be $m = 1M_\odot$.

b. Show that the mean rate of strong encounters per star is

$$\frac{\mathrm{d}n_{\mathrm{enc}}}{\mathrm{d}t} = 4\pi G^2 n m^2 V^{-3} \tag{6.2.2}$$

Hence, demonstrate that the Sun is unlikely to have experienced a strong encounter in its lifetime.

From the above, it follows that most stellar encounters are of the *weak* type. This means that the velocity change of a star, during any one encounter, is typically small ($v_\perp \ll v_{||}$). It is the cumulative effect of many *distant* encounters that will, eventually, be important. For further calculations, we will thus evaluate the forces and accelerations as if the first star continues moving along the original path and the second star remains stationary.

c. Under these assumptions, show that the acceleration of the star perpendicular to $v_{\|}$, integrated over all positions along the path, produces a perpendicular velocity

$$v_{\perp} = 2\frac{Gm}{\beta v_{\|}} \tag{6.2.3}$$

You may find the following integral useful:

$$\int \frac{\mathrm{d}x}{X\sqrt{X}} = 2\frac{2ax+b}{\Delta\sqrt{X}} \tag{6.2.4}$$

where $X \equiv ax^2 + bx + c$ and $\Delta = 4ac - b^2$.

For relative velocities $V \sim v_{\|}$ and stellar density n, the number of encounters with impact parameter between β and $\beta + \mathrm{d}\beta$ in a small time step $\mathrm{d}t$ will be

$$\mathrm{d}^2 N_{\mathrm{enc}} = 2\pi\beta n V \,\mathrm{d}\beta\,\mathrm{d}t \tag{6.2.5}$$

Since the encounters may occur in random directions, the total effect of many encounters (ΔV) is found by adding the contributions of each encounter (6.2.3) quadratically,

$$\Delta V^2 = \sum v_{\perp}^2 \tag{6.2.6}$$

d. Hence show, by integrating over impact parameters in a range $\beta_{\mathrm{min}} < \beta < \beta_{\mathrm{max}}$, that the total (average) velocity change in a small time step $\mathrm{d}t$ is

$$\langle \mathrm{d}V^2 \rangle = \frac{8\pi G^2 m^2 n}{V} \ln\left(\frac{\beta_{\mathrm{max}}}{\beta_{\mathrm{min}}}\right) \mathrm{d}t \tag{6.2.7}$$

It is not obvious what to pick for β_{min} and β_{max}, but since only the logarithm of the ratio of these two quantities enters in the expression, their exact values are not important. Usually, it is reasonable to assume $\ln\Lambda \equiv \ln\left(\frac{\beta_{\mathrm{max}}}{\beta_{\mathrm{min}}}\right) \approx 10$. The quantity $\ln\Lambda$ is also known as the *Coulomb logarithm*.

Finally, the *two-body relaxation time scale*, t_{relax}, is now defined as the time that it takes for the effect of the cumulative distant encounters to produce a velocity change similar to the average relative velocities of the stars, $\langle \mathrm{d}V^2 \rangle = V^2$.

e. Assuming that the density and average relative velocities are constant in time, show that this is now given as

$$t_{\mathrm{relax}} = \frac{V^3}{8\pi G^2 m^2 n \ln\Lambda} \tag{6.2.8}$$

which is the expression discussed in the lecture.

Formulae and constants

Distance modulus (D in pc):

$$m - M = 5 \log_{10} D - 5$$

Black-body radiation:

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

$$I_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

Radius of the Sun: $R_\odot = 7 \times 10^8$ m

Mass of the Sun: $M_\odot = 2 \times 10^{30}$ kg

1 pc = 3.09×10^{16} m

Planck's constant: $h = 6.626 \times 10^{-34}$ m^2 kg s^{-1}

Boltzmann's constant: $k = 1.38 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}

Gravitational constant: $G = 6.673 \times 10^{-11}$ m^3 kg^{-1} s^{-2}

Werkcollege, Cosmology 2016/2017, Week 7

These are the exercises and hand-in assignment for the 8th week of the course *Cosmology*. Every week, one of the problems provides credit towards the final exam. If at least **10** of these problems are handed in and approved, one problem on the final exam may be skipped. The hand-in assignment for this week is **Problem 7.2** below.

7.1 Rotation of "Spiral Nebulae"

In 1914, V. M. Slipher deduced from spectroscopic observations of the Sombrero galaxy (NGC 4594) a rotational velocity of about 100 km/s (at $20''$ from the nucleus). Slipher had also measured positive radial velocities for many spiral "nebulae", often several hundred km/s.

Around the same time, Adriaan van Maanen compared several images of M101 taken over a period of about 15 years and measured an annual rotation of $0.022''$ at a distance of $5'$ from the centre (meaning that, according to van Maanen's measurement, a point located $5'$ from the centre would move $0.022''$ in a year). Van Maanen's measurement was used by Harlow Shapley in the "great debate" as one argument against the idea that spiral nebulae are external galaxies similar to the Milky Way.

Let us now explore some of the implications of these measurements:

1. Based on van Maanen's measurement, what is the rotation period of M101 (in years)?
2. Shapley had estimated that the Sun is located about 15 kpc from the centre of the Milky Way. If the Sun is orbiting around the centre of the Milky Way with the same period as van Maanen's measurement implied for M101, what would be the speed of the Sun? In km/s? In units of c, the speed of light? Would you agree with Shapley that this is unreasonable?
3. If, on the other hand, M101 rotates as fast as NGC 4594 (100 km/s), what would be the distance of M101? Does this seem more reasonable? Why / why not?

Both Slipher's and van Maanen's observations were extremely challenging at the time. An angle of $0.022''$ is tiny. G. W. Ritchie had already measured two of van Maanen's plates before and found no rotation. The spectroscopic measurements were based on exposures that had to extend over many hours, and not everybody believed Slipher's radial velcities, either.

4. The "plate scale" on the photographs used by van Maanen was about $30''\ \mathrm{mm}^{-1}$. For two observations made 15 years apart, what is the shift measured by van Maanen in mm?
5. If you had been attending the debate and knew what was known then, what would you have concluded about the galactic or extragalactic nature of spiral nebulae?

7.2 Radiation Pressure and Radial Velocities

In the "great debate", neither Shapley nor Curtis had a good explanation for the positive radial velocities of the nebulae. Today we know that this is due to the expansion of the Universe itself, but cosmology was still in its infancy in the 1920s and most people believed in a static Universe. Shapley suggested, somewhat hand-wavingly, that the nebulae might be accelerated by radiation pressure from the Milky Way. However, Henry Norris Russell was quick to demonstrate this cannot plausibly work. In this assignment we examine some of Russell's arguments.

Russell made a few simple assumptions:

1. Masses of the nebulae can be estimated from their rotation, assuming the standard Newtonian formula for circular rotation (but note that, strictly speaking, this assumes a spherically symmetric mass distribution). In 1921, such measurements were available for two nebulae: M31 and NGC 4594.
2. The plane of a nebula is perpendicular to the line-of-sight towards the Milky Way.
3. A nebula absorbs all the radiation from the Milky Way that falls upon it.
4. As seen from a nebula, the Milky Way occupies half the sky.
5. Seen from a nebula, the intensity of the light from the Milky Way is similar to that seen from Earth.
6. The intensity of the Milky Way corresponds to 3.5% of the flux from a 1st magnitude star per square degree (this number came from measurements by the Dutch astronomer Pieter van Rhijn, a student of Kapteyn). Such a star is a factor of $10^{0.4\times(1+26.7)} = 1.2\times10^{11}$ times fainter than the Sun.
7. Two measures of the "radius" of a nebula were considered: 1) an "inner" radius r, containing the majority of the mass, and 2) an "outer" radius R that represents the maximum area on which the radiation pressure would act.

The momentum of a photon (or a collection of photons) with energy E is $p = E/c$. Also, recall that pressure is force per area.

- Start by calculating the radiation pressure from a square degree of the Milky Way, seen from a nebula. Show that this pressure is

$$\mathcal{P} = 2.3\times10^{-14}\frac{L_\odot}{c(1\mathrm{AU})^2}$$

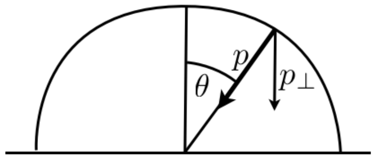
  where 1 AU = 1 astronomical unit = the distance from the Sun to the Earth, $L_\odot$ is the luminosity of the Sun, and c is the speed of light.

- Next, show that the force on the nebula due to radiation pressure from by a whole hemisphere is

$$\mathcal{F} = 7.5\times10^{-10}\frac{D^2R^2L_\odot}{c(1\mathrm{AU})^2}$$

  for distance D.

Hint: For radiation originating somewhere on the hemisphere, only the component of the momentum vector perpendicular to the surface of the nebula ($p_\perp$ in the figure below) contributes to the acceleration. The integral $\int_0^{\pi/2} \sin\theta\cos\theta\,\mathrm{d}\theta = \frac{1}{2}$.



- Finally, show that the acceleration produced by radiation pressure is then

$$A = 7.5 \times 10^{-10} \frac{L_\odot G}{c(1\mathrm{AU})^2} \frac{DR^2}{rv^2}$$

for "inner" radius r, "outer" radius R, circular velocity v at r. G is the gravitational constant.

Some of the assumptions made here (e.g. #4) may seem very unrealistic today, but it is important to keep the context of this calculation in mind. Russell's aim was to examine whether radiation pressure could significantly affect the kinematics of nebulae, given Shapley's view that the Milky Way was very large, and the nebulae all essentially part of the Milky Way.

One of the few nebulae for which the necessary observations were available in 1921 was the "Sombrero galaxy", NGC 4594. NGC 4594 has a radial velocity of +1000 km/s. For r and R, values of $r = 150''$ and $R = 210''$ may be assumed, as well as a rotational velocity of $v = 415$ km/s. The distance was very uncertain, but Russell assumed a distance of 1.43 Mpc or 4.4×10^{22} m.

- Under the above assumptions, calculate the current acceleration of the Sombrero galaxy due to radiation pressure
- If the acceleration had remained constant, and the Sombrero were initially at rest, how long would it have taken to accelerate to the current radial velocity?
- How far would the Sombrero have moved in this time?

Of course, the calculation above is extremely simplified. Which effects have been ignored? How would the calculation change (qualitatively) if these were included?

7.3 Hot gas in dark matter halos

In the classical picture, gas is shock-heated as it falls into dark matter halos and must cool before it can form stars. The rate at which the gas can cool is very sensitive to the composition, because gas that is enriched in heavy elements can cool more efficiently via a large number of atomic line transitions.

Recall that the r.m.s. velocity of particles in a gas with temperature T is given by

$$v_{\rm rms} = \sqrt{\frac{3kT}{\mu}} \tag{7.3.1}$$

where μ is the mean molecular weight, $\mu \approx 10^{-27}$ kg for a highly ionized plasma of typical composition and k is Boltzmann's constant, $k = 1.38 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}.

- Show that we may expect the temperature of the hot gas to be related to the observed line-of-sight velocity dispersion as

$$T = 72 \times 10^6 \left(\frac{\sigma_{1D}}{1000\,{\rm km\,s^{-1}}} \right)^2 {\rm K} \tag{7.3.2}$$

Formulae and constants

Distance modulus (D in pc):

$$m - M = 5 \log_{10} D - 5$$

Black-body radiation:

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

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Werkcollege, Cosmology 2016/2017, Week 12

These are the exercises and hand-in assignment for the 12th week of the course *Cosmology*. Every week, one of the problems provides credit towards the final exam. If at least **10** of these problems are handed in and approved, one problem on the final exam may be skipped. The hand-in assignment for this week is **Problem 12.2** below.

12.1 Cosmological surface brightness dimming (Wed)

In astronomy, the *luminosity* L of a source is the energy output per unit time (e.g. measured in W), the *flux* is the energy passing through a surface of unit area per unit time (e.g. in units of W m^{-2}) and the *intensity* I of radiation is the flux per unit solid angle (W m^{-2} sr^{-1}). It is straight forward to show that the intensity is distance-independent in standard Euclidian geometry, as long as there is no absorbing material between the source and the observer.

- Using the definitions of angular diameter- and luminosity distance, show that the intensity of a source decreases with redshift as

$$I(z) = I_0(1+z)^{-4} \tag{12.1.1}$$

12.2 Cosmological distances (adapted from Reexam 2013/2014) (Wed)

Recall that a line element in the Friedman-Robertson-Walker metric may be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{a^2(t)}{c^2}\left[\mathrm{d}r^2 + \mathcal{R}^2\sin^2(r/\mathcal{R})(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] \tag{12.2.1}$$

for scale factor a, co-moving radial coordinate r, and radius of curvature $\mathcal{R}$.

We have seen that it is useful to define the *angular diameter distance*, D_A, as

$$D_A = \frac{D}{1+z} \tag{12.2.2}$$

for distance measure

$$D = \mathcal{R}\sin(r/\mathcal{R}). \tag{12.2.3}$$

With this definition, we then have following relation between the length dl of a standard rod, oriented perpendicular to the line-of-sight, the apparent angular size of the rod dθ, and D_A:

$$\mathrm{d}l = D_A\mathrm{d}\theta \tag{12.2.4}$$

which is similar to the usual Euclidian relation.

The general expression for the comoving radial coordinate, r, is

$$r = \int_{t_1}^{t_0}\frac{c}{a(t)}\mathrm{d}t \tag{12.2.5}$$

for light emitted from a source at $t = t_1$ and received by an observer at $t = t_0$. In general, this expression must be integrated numerically, although analytic solutions are possible in some cases. Here we explore one such case, the *Einstein-de Sitter Universe*.

In an *Einstein-de Sitter Universe*, $\Omega_0 = 1$ and $\Omega_\Lambda = 0$. For this particular case, the cosmic time t, the Hubble constant H_0, and the scale factor $a(t)$ are related as:

$$a(t) = \left(\frac{3H_0 t}{2}\right)^{2/3} \tag{12.2.6}$$

a. Show that, for an Einstein-de Sitter Universe, the comoving radial coordinate r and the redshift z are related as

$$r = \frac{2c}{H_0}\left(1 - (1+z)^{-1/2}\right) \tag{12.2.7}$$

Hint: The following integral may come in handy:

$$\int_0^a (1+x)^{-3/2}\,\mathrm{d}x = 2\left(1 - (1+a)^{-1/2}\right) \tag{12.2.8}$$

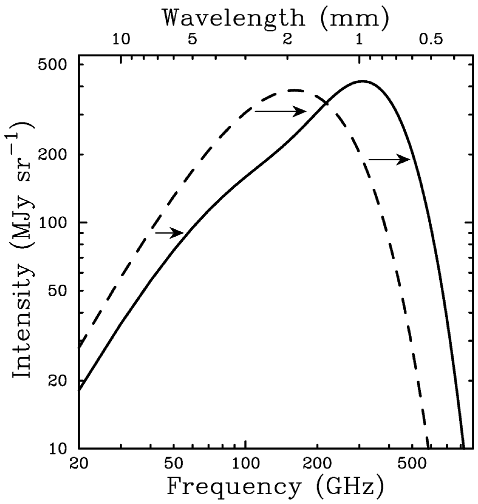
b. Show that D_A has an extremum at $z = 5/4$ in the Einstein-de Sitter Universe, and argue that this must be a maximum. What does this imply for the apparent sizes of objects (of a given linear size) as a function of redshift?

12.3 The Sunyaev-Zeldovich effect (Thu)

In the Sunyaev-Zeldovich effect, CMB photons are inverse Compton scattered to higher energies when passing through hot gas in galaxy clusters. The energy increment is

$$\Delta E_\nu / E_\nu = y \tag{12.3.1}$$

where y is the Compton optical depth. This is illustrated schematically in the figure below:



At low frequencies ($h\nu \ll kT$), the CMB black-body spectrum can be approximated by the Rayleigh-Jeans formula,

$$I_\nu \approx \frac{2\nu^2 kT}{c^2} \tag{12.3.2}$$

a. Convince yourself that an energy boost of the form (12.3.1) corresponds to a purely horizontal shift of the CMB spectrum (when plotted as I_ν, i.e. specific intensity per frequency interval).

b. Then show that in the Rayleigh-Jeans limit, the decrease in the observed intensity is

$$\Delta I_\nu / I_\nu = -2y \tag{12.3.3}$$

Formulae and constants

Distance modulus (D in pc):

$$m - M = 5\log_{10} D - 5$$

Black-body radiation:

$$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT} - 1}$$

$$I_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}$$

Radius of the Sun: $R_\odot = 7 \times 10^8$ m

Mass of the Sun: $M_\odot = 2 \times 10^{30}$ kg

1 pc = 3.09×10^{16} m

Planck's constant: $h = 6.626 \times 10^{-34}$ m^2 kg s^{-1}

Boltzmann's constant: $k = 1.38 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}

Gravitational constant: $G = 6.673 \times 10^{-11}$ m^3 kg^{-1} s^{-2}

Werkcollege, Cosmology 2016/2017, Week 13

These are the exercises and hand-in assignment for the 13th week of the course *Cosmology*. The hand-in assignment for this week is **Problem 13.5** below.

13.1 Gravitational microlensing (Wed)

In 1986, Bohdan Paczyński suggested that dark matter in the form of massive compact halo objects (MACHOs) would be detectable due to gravitational lensing of distant stars. Recall the expression for the angular radius of the Einstein ring:

$$\theta_E^2 = \frac{4GM}{c^2}\left(\frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}D_{\mathrm{L}}}\right) \tag{13.1.1}$$

where M is the mass of the lensing object, D_{LS} is the distance from the lens to the source, and D_{S} and D_{L} are the distances from the observer to the source and lens, respectively.

- A requirement for significant amplification of the source is that it is smaller than the Einstein radius of the lens. Calculate the minimum detectable lensing mass, assuming that the lensing objects are at a typical distance of 10 kpc and that the background stars are solar-type stars in the Large Magellanic Cloud at a distance of 50 kpc.
- Show that, for a population of lenses with a uniform spatial distribution of (mass) density ρ extending all the way to the source population, the optical depth is

$$\tau = \left(\frac{2\pi}{3}\right)\left(\frac{G\rho}{c^2}\right)D_S^2$$

- Show that, if the population of lenses forms a self-gravitating system extending all the way to the source population, then the optical depth depends only on the velocity dispersion σ of the system:

$$\tau \approx \sigma^2/c^2$$

You will need to the following quantities:
$1\ R_\odot = 7 \times 10^8$ m
1 pc $= 3.08 \times 10^{16}$ m
$1\ M_\odot = 2 \times 10^{30}$ kg
$G = 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}

13.2 The flatness problem (Thu)

In this assignment we explore the evolution of the density parameter for matter, Ω_M, with redshift/scale factor. According to the CMB measurements by the Planck satellite, the *current* value is $\Omega_{M,0} = 0.31$ while the dark energy density parameter is $\Omega_{\Lambda,0} = 0.69$, making the Universe exactly flat with $\Omega = 1.0$.

- Recall that the critical density, at any epoch, is defined as

$$\rho_c = 3H^2/8\pi G \tag{13.2.1}$$

  and the time derivative of the scale factor is given by the Friedman equation,

$$\dot{a} = H_0\left[\Omega_{M,0}(1/a - 1) + \Omega_{\Lambda,0}(a^2 - 1) + 1\right]^{1/2} \tag{13.2.2}$$

  Now show that regardless of the present-day values of $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$, the Universe approached an Einstein-de Sitter Universe with $\Omega_M = 1$ at high redshift.

- If the matter density is currently $\Omega_{M,0} = 0.3$, then how much did it deviate from unity at $z = 1000$?

This is the *flatness* problem: Why is the current value of Ω_M close to, but not exactly unity? It requires an exceedingly accurate degree of fine-tuning to produce the tiny departure from $\Omega_M = 1$ at high redshifts that result in a present-day Universe whose density parameter is neither very different from, nor exactly equal to unity.

From a practical perspective, however, it is very convenient that the Universe behaved as an Einstein-de Sitter Universe until relatively recently (in cosmological terms). In terms of structure formation, the regime of linear grown occurred under conditions where the density was very close to the critical value and the Ω_Λ term negligible. At later epochs this is no longer the case, but since the non-linear regime has to be treated numerically in any case the departures from the Einstein-de Sitter Universe do not represent a very serious extra complication.

13.3 Parametric solutions to Friedman's equation (Thu)

Show that the parametric solutions

$$a(\theta) = \frac{\Omega_{M,0}}{2(\Omega_{M,0} - 1)}(1 - \cos\theta) \tag{13.3.1}$$

$$t(\theta) = \frac{\Omega_{M,0}}{2H_0(\Omega_{M,0} - 1)^{3/2}}(\theta - \sin\theta) \tag{13.3.2}$$

satisfy the Friedman equation (13.3.2) for $\Omega_{\Lambda,0} = 0$ and $\Omega_{M,0} > 1$.

13.4 Tophat model (Thu)

In the "tophat" model for the evolution of overdensities we consider each overdensity as a "mini-Universe" with density $\Omega'_0 > 1$ that evolves in a "background Universe" with $\Omega_0 = 1$. Hence, the scale factor of the background Universe evolves as

$$a = \left(\frac{3H_0 t}{2}\right)^{2/3} \tag{13.4.1}$$

while the overdensities evolve according to the parametric solutions in Problem 13.3.

- Show that the density contrast of an overdensity, once it has reached virial equilibrium, is

$$\rho_{\text{vir}}/\rho_0 \approx 150 \tag{13.4.2}$$

13.5 The Press-Schechter mass function (hand-in)

In the lecture we saw how a few basic assumptions lead to a simple analytical formula that provides a remarkably good description of the mass function of bound structures in the Universe:

1. The Universe "initially" (i.e. shortly after the epoch of reionization) consists of particles that are distributed randomly. The variance on the mass within a given volume V is, in this case,

$$\Sigma_V^2 = \sigma^2 V \tag{13.5.1}$$

   where σ^2 is the variance per unit volume.

2. The distribution of overdensities $P(\Delta, V)$ is Gaussian with variance given by (13.5.1)

3. The fluctuations are initially small and grow linearly until they reach a critical value, Σ_{crit}, at which point they immediately collapse and virialize.

These assumptions lead to a mass function for bound fluctuations of the form

$$\frac{\mathrm{d}N}{\mathrm{d}M} \propto M^{-3/2} \exp(-M/M^\star) \tag{13.5.2}$$

where $M^\star \propto a^2$ for scale factor a.

A more general result may be obtained by relaxing the assumption (13.5.1).

- Suppose that the variance follows a relation of the form

$$\Sigma_V^2 = \sigma^2 V^{2\alpha} \tag{13.5.3}$$

  Then, following the same reasoning that led to (13.5.2) (see the lecture viewgraphs), show that the more general mass function has the form

$$\frac{\mathrm{d}N}{\mathrm{d}M} \propto M^{-1-\alpha} \exp\left(-\left[\frac{M}{M^\star}\right]^{2(1-\alpha)}\right) \tag{13.5.4}$$

  with

$$M^\star \propto a^{1/(1-\alpha)} \tag{13.5.5}$$

  The relation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{erfc}\left(a\xi^b\right) = -\frac{2ab\exp\left(-a^2\xi^{2b}\right)\xi^{b-1}}{\sqrt{\pi}} \tag{13.5.6}$$

  might be useful.

Werkcollege, Cosmology 2016/2017, Week 14

These are the exercises and hand-in assignment for the 14th week of the course *Cosmology*. The hand-in assignment for this week is **Problem 14.4** below.

14.1 Decaying potentials

We have seen in earlier lectures that small density perturbations in a Universe dominated by pressure-less dark matter grow linearly with the scale factor, i.e.,

$$\frac{\delta\rho}{\rho} \propto a \tag{14.1.1}$$

Here we examine the evolution of perturbations of the underlying potential, Ψ. Let us assume for simplicity that the perturbations are spherically symmetric.

- Suppose that a test particle is located at the outer "boundary" of a perturbation with co-moving radius r. Use the classical definition of the gravitational potential to show that, in the linear regime, the perturbation of the potential $\delta\Psi$ remains constant as the scale factor increases.
- Also show that, if the perturbations grow more slowly than a, the perturbation of the potential will decay as the scale factor increases.

14.2 Newtonian equivalence of metric perturbations

(From Dodelson, Exercise 3, Chapter 4)

The metric for a particle travelling in the presence of a gravitational field is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{00} = -2\phi$ where ϕ is the Newtonian gravitational potential; $h_{i0} = 0$ and $h_{ij} = -2\phi\delta_{ij}$:

$$g_{\mu\nu} = \begin{pmatrix} -1-2\phi & 0 & 0 & 0 \\ 0 & 1-2\phi & 0 & 0 \\ 0 & 0 & 1-2\phi & 0 \\ 0 & 0 & 0 & 1-2\phi \end{pmatrix} \tag{14.2.1}$$

- Show that $\Gamma^i{}_{00} = \delta^{ij}\partial\phi/\partial x^j$
- Show that the space components of the geodesic equation lead to $\mathrm{d}^2x^i/\mathrm{d}t^2 = -\delta^{ij}\mathrm{d}\phi/\mathrm{d}x^j$ in agreement with Newtonian theory. Use the fact that the particle is non-relativistic so $P^0 \gg P^i$.

14.3 Four-momentum of photons in perturbed FRW metric

We adopt the perturbed version of the FRW metric as follows:

$$g_{\mu\nu} = \begin{pmatrix} -1-2\Psi(x,t) & 0 & 0 & 0 \\ 0 & a^2[1+2\Phi(x,t)] & 0 & 0 \\ 0 & 0 & a^2[1+2\Phi(x,t)] & 0 \\ 0 & 0 & 0 & a^2[1+2\Phi(x,t)] \end{pmatrix} \tag{14.3.1}$$

In the lecture we found that, to first order, the 0th component of the energy-momentum four-vector can be written as

$$P^0 \simeq p(1-\Psi) \tag{14.3.2}$$

where

$$p \equiv g_{ij}P^iP^j \tag{14.3.3}$$

- Now show that the other components of the momentum four-vector can be written as

$$P^i \simeq p\hat{p}^i \frac{1-\Phi}{a} \tag{14.3.4}$$

where $\hat{p}$ is the unit vector parallel to p.

14.4 The momentum time derivative

We have expanded the left-hand side of the Boltzmann equation in terms of the partial derivatives with respect to t, x and p as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \cdot \frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial f}{\partial \hat{p}^i} \cdot \frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t} \tag{14.4.1}$$

Using the definitions of p and $\hat{p}$, and keeping only first-order terms, we saw how this reduces to

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a} \cdot \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} \tag{14.4.2}$$

The momentum term is non-trivial and requires a bit more work. So let's get started! First, we use the 0th component of the geodesic equation:

$$\frac{\mathrm{d}^2 x^0}{\mathrm{d}\lambda^2} = -\Gamma^0{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} \tag{14.4.3}$$

- Show that, in first instance, Eq. (14.4.3) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p(1-\Psi)\right] = -\Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+\Psi) \tag{14.4.4}$$

  (i.e., Eq. 4.23 in Dodelson's book). *Hint:* as usual, keep only first order terms (linear in Ψ)!

- Next, expand out the time derivative on the left-hand side and show that this leads to

$$\frac{\mathrm{d}p}{\mathrm{d}t}(1-\Psi) = p\frac{\mathrm{d}\Psi}{\mathrm{d}t} - \Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+\Psi) \tag{14.4.5}$$

  (i.e. Eq. 4.24 in the book)

- Now, multiply by $(1+\Psi)$ to find Eq. (4.25):

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p\left(\frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right) - \Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+2\Psi) \tag{14.4.6}$$

- Finally, evaluate the Christoffel symbol and show that

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -p\left(H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right) \tag{14.4.7}$$

  Hint: See p. 91–92 in Dodelson's book.

We have now finished manipulating the left-hand side of the Boltzmann equation for photons:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a} \cdot \frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left(H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right) \tag{14.4.8}$$

14.5 First order terms of the Boltzmann equation for photons

- Demonstrate that the *first-order* terms in the left-hand side of the Boltzmann equation for photons (Equation (4.40) in Dodelson's book),

$$\left.\frac{df}{dt}\right|_1 = -p\frac{\partial}{\partial t}\left(\frac{\partial f^{(0)}}{\partial p}\Theta\right) - p\frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}\left(\frac{\partial f^{(0)}}{\partial p}\right) + Hp\Theta\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right) - p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] \quad (14.5.1)$$

  follow from expression (14.4.8), combined with the perturbed expansion of the photon distribution,

$$f = f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta \quad (14.5.2)$$

- The next equation in the book, (4.41), says that the first of these terms can be written as

$$-p\frac{\partial}{\partial t}\left(\frac{\partial f^{(0)}}{\partial p}\Theta\right) = -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial t} - p\Theta\frac{dT}{dt}\frac{\partial^2 f^{(0)}}{\partial T\partial p} \quad (14.5.3)$$

$$= -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial t} + p\Theta\frac{dT/dt}{T}\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right) \quad (14.5.4)$$

  Show that the second term in Eq. (14.5.4) does indeed cancel the third term in Eq. (14.5.1) so that the first-order terms of the left-hand side of the Boltzmann equation for photons become

$$\left.\frac{df}{dt}\right|_1 = -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] \quad (14.5.5)$$

14.6 Exercise 5, Chapter 4

Suppose we started chapter 4 by writing

$$\frac{df}{d\lambda} = C' \quad (14.6.1)$$

Change from this form to the one in Eq. (4.1) (with df/dt *on the left). How is the collision term here,* C' *related to* C *in Eq. (4.1)? Argue that the first-order perturbations in the factor relating the two collision terms can be dropped since the collision terms themselves are first-order.*

14.7 The Einstein tensor in the perturbed FRW metric

To calculate the perturbations of the metric, Ψ and Φ, given the inhomogeneities in the distribution of matter and radiation, we need Einstein's field equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{14.7.1}$$

with the Einstein tensor given by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} \tag{14.7.2}$$

Specifically, we choose the $(0, 0)$ component, with

$$G^0{}_0 = g^{0i} G_{i0} = (-1 + 2\Psi) R_{00} - \frac{\mathcal{R}}{2} \tag{14.7.3}$$

for Ricci tensor

$$R_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} + \Gamma^\alpha{}_{\beta\alpha} \Gamma^\beta{}_{\mu\nu} - \Gamma^\alpha{}_{\beta\nu} \Gamma^\beta{}_{\mu\alpha} \tag{14.7.4}$$

and Ricci scalar $\mathcal{R} = g^{\mu\nu} R_{\mu\nu}$.

To calculate $\mathcal{R}$, we need all elements of $R_{\mu\nu}$ and thus the complete set of Christoffel symbols. Here, we calculate a few of them.

- Show the following relations (as usual, to first order in the perturbations of the metric):

$$\Gamma^0{}_{00} \simeq \Psi_{,0} \tag{14.7.5}$$

$$\Gamma^0{}_{i0} \simeq i k_i \tilde{\Psi} \tag{14.7.6}$$

$$\Gamma^0{}_{ij} \simeq \delta_{ij} a^2 \left[H + 2H(\Phi - \Psi) + \Phi_{,0} \right] \tag{14.7.7}$$

  where the tilde denotes the transformation to Fourier space.

Werkcollege, Cosmology 2017/2016, Week 15

These are the exercises for the 15th week of the course *Cosmology*. The hand-in assignment for this week is **Problem 15.3** below.

15.1 Momenta of the photon perturbations

Show that

$$\int_{-1}^{1} \mathrm{d}\mu\mu^2\Theta(\mu) = \frac{2}{3}\Theta_0 - \frac{4}{3}\Theta_2 \tag{15.1.1}$$

15.2 From inhomogeneities to anisotropies (I)

In this exercise we fill in some of the details in the calculation of *anisotropies* in the observed temperature distribution on the sky from the *inhomogeneities* around recombination.

We start, once again, from the Boltzmann equation for photons:

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu\mathbf{v}_b\right] \tag{15.2.1}$$

We are now, of course, interested in the high-order moments Θ_l that correspond to the (small) variations in the CMB temperature observed *today*, from our viewpoint at $\eta = \eta_0$. We start by subtracting $\dot{\tau}\Theta$ from both sides:

$$\begin{aligned} \dot{\Theta} + ik\mu\Theta - \dot{\tau}\Theta &= -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu\mathbf{v}_b\right] - \dot{\tau}\Theta & (15.2.2)\\ \dot{\Theta} + (ik\mu - \dot{\tau})\Theta &= -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 + \mu\mathbf{v}_b\right] & (15.2.3) \end{aligned}$$

- Verify that the left-hand side can be rewritten as

$$e^{-ik\mu\eta+\tau}\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\Theta e^{ik\mu\eta-\tau}\right] = \dot{\Theta} + (ik\mu - \dot{\tau})\Theta \tag{15.2.4}$$

We define the right-hand side as the *source function* (borrowing terminology from the theory of radiative transfer in stellar atmospheres, which shares many aspects with this calculation),

$$\tilde{S} \equiv -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 + \mu\mathbf{v}_b\right] \tag{15.2.5}$$

- Then show that the perturbations at conformal time η_0 are related to those at η_{init} as

$$\Theta(\eta_0) = \Theta(\eta_{\mathrm{init}})e^{ik\mu\eta_{\mathrm{init}}-\tau}e^{-ik\mu\eta_0+\tau} + e^{-ik\mu\eta_0+\tau}\int_{\eta_{\mathrm{init}}}^{\eta_0} \mathrm{d}\eta\tilde{S}\, e^{ik\mu\eta-\tau} \tag{15.2.6}$$

  and, if η_0 is today and η_{init} is long before recombination

$$\Theta(\eta_0) \simeq \int_{\eta_{\mathrm{init}}}^{\eta_0} \mathrm{d}\eta\tilde{S}\, e^{ik\mu(\eta-\eta_0)-\tau(\eta)} \tag{15.2.7}$$

15.3 From inhomogeneities to anisotropies (II)

We now need to calculate the multipole moments, defined as

$$\Theta_l \equiv \frac{1}{(-i)^l}\int_{-1}^{1}\frac{\mathrm{d}\mu}{2}\mathcal{P}_l(\mu)\Theta(\mu,\eta_0) \tag{15.3.1}$$

with $\Theta(\mu,\eta_0)$ given by

$$\Theta(\mu,\eta_0) = \int_0^{\eta_0}\mathrm{d}\eta\tilde{S}\,e^{ik\mu(\eta-\eta_0)-\tau(\eta)} \tag{15.3.2}$$

If $\tilde{S}$ did not depend on μ, this would be easy since

$$\int_{-1}^{1}\frac{\mathrm{d}\mu}{2}\mathcal{P}_l(\mu)e^{ik\mu(\eta-\eta_0)} = \frac{1}{(-i)^l}j_l[k(\eta-\eta_0)] \tag{15.3.3}$$

where j_l is the spherical Bessel function of order l. So let us split $\tilde{S}$ into two parts,

$$\tilde{S}_1 \equiv -\dot{\Phi} - \dot{\tau}\Theta_0 \tag{15.3.4}$$

$$\tilde{S}_2 \equiv -\mu(ik\Psi + \dot{\tau}v_b) \tag{15.3.5}$$

where $\tilde{S}_2$ depends on μ and $\tilde{S}_1$ does not.

- Evaluate the part of Θ_l involving $\tilde{S}_1$ (call it $\Theta_{l,1}$) and show that

$$\Theta_{l,1} = (-1)^l\int_0^{\eta_0}\mathrm{d}\eta e^{-\tau}\left(-\dot{\Phi} - \dot{\tau}\Theta_0\right)j_l[k(\eta-\eta_0)] \tag{15.3.6}$$

Next, we make use of the fact that $\tilde{S}_2$ appears multiplied by $e^{ik\mu(\eta-\eta_0)}$.

- Demonstrate that μ, when appearing in this context, can be replaced by

$$\mu \to \frac{1}{ik}\frac{\mathrm{d}}{\mathrm{d}\eta} \tag{15.3.7}$$

- Show that the integral involving $\tilde{S}_2$ evaluates to

$$\int_0^{\eta_0}\mathrm{d}\eta\tilde{S}_2e^{ik\mu(\eta-\eta_0)-\tau(\eta)} = \mathrm{Const} - \int_0^{\eta_0}\mathrm{d}\eta\,e^{ik\mu(\eta-\eta_0)}\frac{\mathrm{d}}{\mathrm{d}\eta}\left[e^{-\tau(\eta)}\left(-\Psi + \frac{i\dot{\tau}v_b}{k}\right)\right] \tag{15.3.8}$$

  where Const is independent of μ (so irrelevant when computing the Θ_l). Use that $\tau(0) \gg 1$ so that $e^{-\tau(0)} \simeq 0$. In case you forgot, here is the formula for integrating by parts:

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int v(x)u'(x)\mathrm{d}x \tag{15.3.9}$$

- Then show that the contribution of $\tilde{S}_2$ to the multipole moments is

$$\Theta_{l,2} = (-1)^l\int_0^{\eta_0}\mathrm{d}\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\left[e^{-\tau(\eta)}\left(\Psi - \frac{i\dot{\tau}v_b}{k}\right)\right]j_l[k(\eta-\eta_0)] \tag{15.3.10}$$

- Finally, introducing the visibility function $g(\eta) = -\dot{\tau}e^{-\tau}$, verify that the following two forms of the source function S are equivalent:

$$\begin{aligned} S(k,\eta) &= e^{-\tau}(-\dot{\Phi} - \dot{\tau}\Theta_0) + \frac{\mathrm{d}}{\mathrm{d}\eta}\left[e^{-\tau(\eta)}\left(\Psi - \frac{i\dot{\tau}v_b}{k}\right)\right] && (15.3.11) \\ &= g(\eta)[\Theta_0 + \Psi] + \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{iv_b g(\eta)}{k}\right) + e^{-\tau}\left[\dot{\Psi}(k,\eta) - \dot{\Phi}(k,\eta)\right] && (15.3.12) \end{aligned}$$

  The latter form shows more clearly that the observed CMB anisotropies contain terms of three types: 1) The monopole of the temperature perturbations combined with metric perturbations around the recombination, 2) The bulk velocity (which is coupled to the temperature dipole), also around recombination, 3) Temporal variations in the metric perturbations (i.e., the potential) along the entire line-of-sight.

Solutions, Cosmology 2016/2017, Week 4

4.1 F_λ and F_ν

If we look at a small part of the spectrum $\mathrm{d}\lambda$, then the flux is

$$\mathrm{d}F = F_\lambda \mathrm{d}\lambda \tag{4.1.13}$$

Similarly,

$$\mathrm{d}F = F_\nu \mathrm{d}\nu \tag{4.1.14}$$

so

$$F_\lambda \mathrm{d}\lambda = F_\nu \mathrm{d}\nu \tag{4.1.15}$$

$$\frac{F_\nu}{F_\lambda} = \frac{\mathrm{d}\lambda}{\mathrm{d}\nu} \tag{4.1.16}$$

Using

$$\lambda = c/\nu \tag{4.1.17}$$

we have

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\nu} = -\frac{c}{\nu^2} \tag{4.1.18}$$

so

$$\frac{F_\nu}{F_\lambda} = \frac{c}{\nu^2} = \frac{\lambda^2}{c} \tag{4.1.19}$$

as desired (F_ν and F_λ must both be positive, of course).

4.2 Flux of astronomical objects

1. For $\lambda = 5500$ Å $= 5500 \times 10^{-10}$ m and $T = 9900$ K we get $I_\lambda = 1.81 \times 10^{14}$ W m^{-2} m^{-1} sr^{-1}.

2. Multiply I_λ by $\Delta\lambda_V = 900 \times 10^{-10}$ m $\Rightarrow I_V = 1.63 \times 10^7$ W m^{-2} sr^{-1}

3. There are (at least) two ways to do this:

   a. V-band luminosity of Sirius: $L_V = 4\pi R^2(\pi I_V) = 9.55 \times 10^{26}$ W (factor πI_V comes from integrating I_V over angles from 0 to $\pi/2$ with respect to the normal: $P_V = \int_0^{\pi/2} 2\pi \sin\theta \cos\theta I_V d\theta = \pi I_V$). Then the flux measured at Earth $= F_V = \frac{L_V}{4\pi D^2} = 1.18 \times 10^{-8}$ W m^{-2}.

   b. Since intensity is distance independent, we can also obtain F_V by integrating over the disk of Sirius as seen from Earth: $F_V = I_V \Omega$, where $\Omega = \pi(R/D)^2$ (we have assumed that I_V is constant across the surface, as it will be for a pure black-body). This again yields $F_V = 1.18 \times 10^{-8}$ W m^{-2}.

4. Number of photons per square meter: $N_V = F_V/E_V$ where E_V is the energy of a V-band photon $= h\nu_V = hc/\lambda_V$. We get $N_V = 3.27 \times 10^{10}$ m^{-2} s^{-1}. Number of photons entering HST aperture: multiply by $\pi(1.2\text{m})^2 \Rightarrow 1.48 \times 10^{11}$ s^{-1}.

4.3 Flux, Magnitude and Surface Brightness

1. Since Vega has $m_V = 0$, we have

$$m_V(1\,\mathrm{Jy}) = -2.5\log_{10}\frac{1\,\mathrm{Jy}}{F_{\mathrm{Vega}}} = 8.9 \tag{4.3.20}$$

2. From the definition of magnitudes,

$$m_1 - m_2 = -2.5\log_{10}(F_1/F_2) \tag{4.3.21}$$

we have

$$F_1/F_2 = 10^{-0.4\times(m_1-m_2)} \tag{4.3.22}$$

so again, using Vega as reference, we have

$$F(m=6) = F_{\mathrm{Vega}} \times 10^{-0.4\times 6} = 1.43\times 10^{-25}\,\mathrm{W\,m^{-2}\,Hz^{-1}} \approx 14\,\mathrm{Jy} \tag{4.3.23}$$

3. The flux received from a 6th magnitude star, relative to one square arcsecond of sky background, is

$$F_{6\mathrm{mag}}/F_{\mathrm{bkg}} = 10^{-0.4\times(6-22)} = 2.5\times 10^6 \tag{4.3.24}$$

We thus need an area of 2.51×10^6 arcsec^2 = 698 arcmin^2 (R = 15 arcmin, similar to full Moon or Sun)

Solutions, Cosmology 2015/2016, Week 5

5.1 Distance and distance modulus

Can be shown in a couple of ways.
1) The distance modulus is

$$m - M = 5\log D/10\,\mathrm{pc}$$

So for distances D_1 and D_2, the distance moduli are

$$(m - M)_i = 5\log D_i/10\mathrm{pc}$$

and the difference is

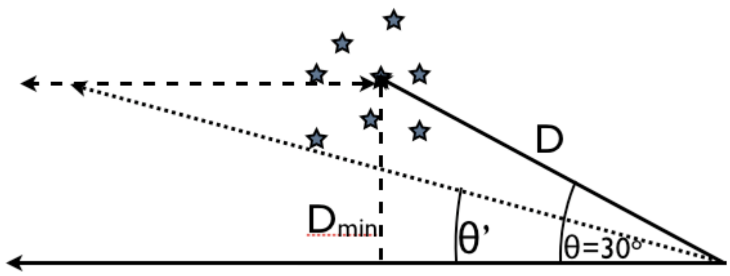
$$(m - M)_1 - (m - M)_2 = 5\log D_1/D_2$$

so if $(m - M)_1 - (m - M)_2 = 0.1$, then the ratio $D_1/D_2 \approx 1.05$, so a 5% error on the distance.
2) We can also use standard error propagation:

$$\begin{aligned}\delta(m - M) &= \frac{\partial(5\log D/10\mathrm{pc})}{\partial D}\delta D \\ &= \frac{5}{\ln 10}\frac{\delta D}{D}\end{aligned}$$

Hence, $\delta D/D = \delta(m - M)\frac{1}{5}\ln 10 \approx 0.5\delta(m - M)$

5.2 Moving cluster method



1. Distance from moving cluster method: $D = \frac{v_r \tan\theta}{\mu}$. Insert $v_r = 40$ km s^{-1} and $\mu = 0.110''$ yr^{-1} yields $D = 44.4$ pc.

2. See above - min dist $D_{\min} = \sin 30° D = 0.5D = 22.2$ pc.

3. Closest passage: Distance travelled = $D_c = D\cos\theta$. Time $T_c = D_c/v = D\cos\theta/v$. Space velocity $v = v_t/\sin\theta = D\mu/\sin\theta$. Then $T_c = (D\cos\theta)/(D\mu/\sin\theta) = \cos\theta\sin\theta/\mu = 8.1\times10^5$ years ago.

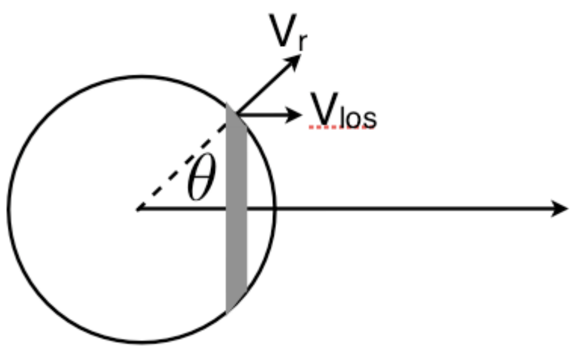
4. Fading by 1 magnitude $\rightarrow$ distance increase by factor $\sqrt{2.512} = 1.585$ to $D' = 70.4$ pc. $\Delta X = D'\cos\theta' - D\cos\theta$. $\Delta T' = \Delta X/v = (D'\cos\theta' - D\cos\theta)/(D\mu/\sin\theta) = (1.585D\cos\theta' - D\cos\theta)/(D\mu/\sin\theta) = \sin\theta(1.585\cos\theta' - \cos\theta)/\mu$. We have $\sin\theta' = D_{\min}/D' = 0.5D/1.585D = 0.315$, i.e. $\Delta T' = 6.0\times10^5$ years.

5.3 Cepheids

1. From the parallax, the distance to δ Cep = 1/3.8 × 10−3 = 263 pc
This gives an absolute magnitude $M_V = 3.8 - (5\log_{10} d - 5) = -3.3$
Using δ Cep to set the absolute zero-point of the period-luminosity relation, we find $-3.3 = -2.5 \times 0.73 + Z_V$, i.e., $Z_V = -1.47$, i.e. $-3.3 = -2.5\log_{10} P - 1.47$.
Compare this with the zero-point for the LMC P-L relation to find the distance modulus $m - M = 17.0 - (-1.47) = 18.47$. This gives the distance, $D = 49$ kpc.

2. The extinction corrected magnitude is $V_0 = 27.1 - 0.1 = 27.0$.
From the P-L relation, we get $M_V = -2.5\log_{10} P - 1.47 = -3.97$
Distance modulus $(m - M)_0 = 27.0 - (-3.97) = 31.0$ so $D = 15.8$ Mpc

5.4 Baade-Wesselink method



In the sketch above, V_r is the expansion velocity of the stellar surface while $V_{\rm los}$ is the component of that velocity directed towards the observer, $V_{\rm los} = V_r \cos\theta$. The observer will see a "ring" with surface area $da = 2\pi \sin\theta \cos\theta d\theta$ and intensity $I(\theta)$ expanding at $V_{\rm los}(\theta)$. The mean line-of-sight velocity integrated over all θ and weighted by $I(\theta)$ is then

$$\langle V_{\rm los} \rangle = \frac{\int I(\theta)\, V_{\rm los}(\theta)\, da}{\int I(\theta)\, da} = \frac{\int_0^{\pi/2} I(\theta) V_r \cos\theta\, 2\pi \sin\theta \cos\theta d\theta}{\int_0^{\pi/2} I(\theta) 2\pi \sin\theta \cos\theta\, d\theta}$$

$$= V_r \frac{\int_0^{\pi/2} I(\theta) \cos^2\theta \sin\theta\, d\theta}{\int_0^{\pi/2} I(\theta) \sin\theta \cos\theta\, d\theta}$$

For constant $I(\theta)$,

$$\langle V_{\rm los} \rangle / V_r = \frac{\int_0^{\pi/2} \cos^2\theta \sin\theta\, d\theta}{\int_0^{\pi/2} \sin\theta \cos\theta\, d\theta}$$

Now substitute $U(x) = \cos x$ and $U'(x) = -\sin x$ so that

$$\int_0^{\pi/2} \cos^2\theta \sin\theta\, d\theta = -\int_0^{\pi/2} U^2(\theta) U'(\theta)\, d\theta = -\int_{U(0)}^{U(\pi/2)} x^2\, dx = \int_1^0 x^2\, dx = \frac{1}{3}$$

Similarly,

$$\int_0^{\pi/2} \sin\theta \cos\theta\, d\theta = \frac{1}{2}$$

so

$$\langle V_{\rm los}\rangle / V_r = \frac{3}{2}$$

For $I(\theta) = I(0)(0.4 + 0.6\cos\theta)$:

$$\langle V_{\rm los}\rangle / V_r = \frac{\int_0^{\pi/2}(0.4 + 0.6\cos\theta)\cos^2\theta\sin\theta\, d\theta}{\int_0^{\pi/2}(0.4 + 0.6\cos\theta)\cos\theta\sin\theta\, d\theta}$$

Making substitutions similar to those above, we get

$$\langle V_{\rm los}\rangle / V_r = \frac{24}{17}$$

5.5 K-corrections

1. K-correction in wavelength units :

$$K = 2.5\log_{10}\frac{\int f(\lambda)S(\lambda){\rm d}\lambda}{\int f[\lambda/(1+z)]S(\lambda){\rm d}\lambda} + 2.5\log_{10}(1+z)$$

This comes from

$$K = 2.5\log_{10}\frac{\int_{\lambda_1}^{\lambda_2} f(\lambda)S(\lambda){\rm d}\lambda}{\int_{\lambda_1/(1+z)}^{\lambda_2/(1+z)} f(\lambda')S[\lambda'(1+z)]{\rm d}\lambda'}$$

and

$$\lambda' = \lambda/(1+z)$$
$${\rm d}\lambda' = {\rm d}\lambda/(1+z)$$

that is

$$\begin{aligned} K &= 2.5\log_{10}\frac{\int_{\lambda_1}^{\lambda_2} f(\lambda)S(\lambda){\rm d}\lambda}{\int_{\lambda_1}^{\lambda_2} f[\lambda/(1+z)]S(\lambda)(1+z)^{-1}{\rm d}\lambda} \\ &= 2.5\log_{10}\frac{\int_{\lambda_1}^{\lambda_2} f(\lambda)S(\lambda){\rm d}\lambda}{\int_{\lambda_1}^{\lambda_2} f[\lambda/(1+z)]S(\lambda){\rm d}\lambda} + 2.5\log_{10}(1+z) \end{aligned}$$

2. In frequency units:

$$K = 2.5\log_{10}\frac{\int_{\nu_1}^{\nu_2} f(\nu)S(\nu){\rm d}\nu}{\int_{\nu_1(1+z)}^{\nu_2(1+z)} f(\nu')S[\nu'/(1+z)]{\rm d}\nu'}$$

Here, we have

$$\nu' = \nu(1+z)$$
$${\rm d}\nu' = {\rm d}\nu(1+z)$$

so that

$$
\begin{aligned}
K &= 2.5\log_{10}\frac{\int_{\nu_1}^{\nu_2} f(\nu)S(\nu)\mathrm{d}\nu}{\int_{\nu_1(1+z)}^{\nu_2(1+z)} f(\nu')S[\nu'/(1+z)]\mathrm{d}\nu'} \\
&= 2.5\log_{10}\frac{\int_{\nu_1}^{\nu_2} f(\nu)S(\nu)\mathrm{d}\nu}{\int_{\nu_1}^{\nu_2} f[\nu(1+z)]S(\nu)\mathrm{d}\nu'}(1+z)\mathrm{d}\nu \\
&= 2.5\log_{10}\frac{\int_{\nu_1}^{\nu_2} f(\nu)S(\nu)\mathrm{d}\nu}{\int_{\nu_1}^{\nu_2} f[\nu(1+z)]S(\nu)\mathrm{d}\nu'}\mathrm{d}\nu - 2.5\log_{10}(1+z)
\end{aligned}
$$

3. For $S(\lambda) \propto \lambda^{\beta}$ and assuming the filter transmission curve is a box function, we get

$$
\begin{aligned}
K &= 2.5\log_{10}\frac{\int_{\lambda_1}^{\lambda_2} \lambda^{\beta}\mathrm{d}\lambda}{\int_{\lambda_1}^{\lambda_2}[\lambda/(1+z)]^{\beta}\mathrm{d}\lambda} + 2.5\log_{10}(1+z) \\
&= 2.5\log_{10}\frac{(\beta+1)^{-1}\left(\lambda_2^{1+\beta}-\lambda_1^{1+\beta}\right)}{(\beta+1)^{-1}(1+z)^{-\beta}\left(\lambda_2^{1+\beta}-\lambda_1^{1+\beta}\right)} + 2.5\log_{10}(1+z) \\
&= 2.5\log_{10}(1+z)^{\beta} + 2.5\log_{10}(1+z) \\
&= 2.5\log_{10}(1+z)^{\beta+1}
\end{aligned}
$$

Solutions, Cosmology 2016/2017, Week 6

6.1 Mass distribution in the Milky Way

a. Circular motion:

$$v_c = \sqrt{\frac{GM}{R}} \tag{6.1.1}$$

so

$$M = \frac{v_c^2 R}{G} \tag{6.1.2}$$

The mass within R is also

$$M = 4\pi \int_0^R r^2 \rho(r) \mathrm{d}r \tag{6.1.3}$$

Differentiating, we find

$$\frac{\mathrm{d}M}{\mathrm{d}R} = 4\pi R^2 \rho(R) = \frac{v_c^2}{G} \tag{6.1.4}$$

since v_c = const and hence

$$\rho(R) = \frac{v_c^2}{4\pi G} R^{-2} \tag{6.1.5}$$

b. We just need to fill in the numbers: $v_c = 200 \times 10^3$ m/s, $R_0 = 8$ kpc $= 2.46 \times 10^{20}$ m. Then $\rho = 7.9 \times 10^{-22}$ kg/m^3 $= 0.012\, M_\odot\, \mathrm{pc}^{-3}$

c. At $z = 0$, we have $\rho_d = 0.08\, M_\odot\, \mathrm{pc}^{-3}$. To find the height where $\rho_d = \rho_h$:

$$0.012 = 0.08 e^{-z/z_{\mathrm{scl}}} \tag{6.1.6}$$

i.e.

$$z = -z_{\mathrm{scl}} \ln(0.012/0.08) = 580\,\mathrm{pc} \tag{6.1.7}$$

6.2 Two-body relaxation

a. The potential energy during the encounter is

$$U(r) = -\frac{Gm^2}{r} \tag{6.2.1}$$

while the kinetic energy is

$$T = \frac{1}{2} m V^2 \tag{6.2.2}$$

Equating the two,

$$\frac{Gm^2}{r} = \frac{1}{2} m V^2 \tag{6.2.3}$$

so

$$\beta = \frac{2Gm}{v_{\|}^2} \tag{6.2.4}$$

b. We find $\beta = 2.65 \times 10^{12}$ m. For stellar density n and velocities V, the number of encounters within a radius β in time t is

$$n_{\mathrm{enc}} = n\pi\beta^2 V t \tag{6.2.5}$$

Inserting β from above,

$$
\begin{aligned}
n_{\rm enc} &= n\pi\left(\frac{2Gm}{V^2}\right)^2 Vt && (6.2.6)\\
&= n\pi 4G^2m^2V^{-3}t && (6.2.7)
\end{aligned}
$$

or

$$t_{\rm enc} = \frac{V^3}{4\pi G^2 n m^2} \qquad (6.2.8)$$

For $V = 10$ km/s, $n = 0.1$ pc and $m = 1M_\odot$, we find $t_{\rm enc} = 1.3 \times 10^{21} s$ or 4.1×10^{13} years. So $\sim 1000\times$ the age of the Sun.

c. The acceleration depends on where exactly the particle is with respect to the deflecting mass. If $l = v_{||}t$ is the distance from the closest encounter, then the total acceleration of the particle is

$$a = \frac{Gm}{r^2} = \frac{Gm}{\beta^2 + l^2} \qquad (6.2.9)$$

The component of this perpendicular to $v_{||}$ is

$$
\begin{aligned}
a_\perp &= a\frac{\beta}{r} = \frac{Gm}{\beta^2 + v_{||}^2 t^2}\frac{\beta}{\sqrt{\beta^2 + v_{||}^2 t^2}} && (6.2.10)\\
&= \frac{Gm\beta}{(\beta^2 + v_{||}^2 t^2)^{3/2}} && (6.2.11)
\end{aligned}
$$

The total velocity change is found by integrating over all l, i.e.

$$
\begin{aligned}
v_\perp &= \int_{-\infty}^{\infty} a_\perp \mathrm{d}t = \int_{-\infty}^{\infty} \frac{Gm\beta}{(\beta^2 + v_{||}^2 t^2)^{3/2}}\mathrm{d}t && (6.2.12)\\
&= 2\frac{Gm}{v_{||}\beta} && (6.2.13)
\end{aligned}
$$

The integral (6.2.12) above may evaluated using Bronshtein et al. (2004), *Handbook of Mathematics*, integral #242, p. 1032:

$$\int \frac{\mathrm{d}x}{X\sqrt{X}} = \frac{2(2ax+b)}{\Delta\sqrt{X}} \qquad (6.2.14)$$

where $X = ax^2 + bx + c$ and $\Delta = 4ac - b^2$. In our case, $a = v_{||}^2$, $b = 0$ and $c = \beta^2$. Then

$$\int \frac{\mathrm{d}t}{(\beta^2 + v_{||}^2 t^2)^{3/2}} = 2\frac{2v_{||}^2 t}{4v_{||}^2\beta^2\sqrt{v_{||}^2 t^2 + \beta^2}} \qquad (6.2.15)$$

For limits $-\infty$ and $+\infty$, we get

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(\beta^2 + v_{||}^2 t^2)^{3/2}} &= \left[\frac{t}{\beta^2 v\,|t|}\right]_{-\infty}^{\infty} && (6.2.16)\\
&= \frac{2}{\beta^2 v_{||}} && (6.2.17)
\end{aligned}
$$

Multiplying by $Gm\beta$, we then have

$$v_\perp = \int_{-\infty}^{\infty} a_\perp \mathrm{d}t = \int_{-\infty}^{\infty} \frac{Gm\beta}{(\beta^2 + v_{||}^2 t^2)^{3/2}}\mathrm{d}t = \frac{2Gm}{\beta v_{||}} \qquad (6.2.18)$$

d. This follows straight forwardly by integrating over all β

$$\begin{aligned}
\mathrm{d}V^2 &= \int_{\beta_{\min}}^{\beta_{\max}} (v_\perp)^2 \mathrm{d}^2 N_{\mathrm{enc}} && (6.2.19)\\
&= \int_{\beta_{\min}}^{\beta_{\max}} \left(2\frac{Gm}{\beta V}\right)^2 2\pi\beta n V\, \mathrm{d}\beta\, \mathrm{d}t && (6.2.20)\\
&= \frac{8\pi G^2 m^2 n}{V} \ln\left(\frac{\beta_{\max}}{\beta_{\min}}\right) \mathrm{d}t && (6.2.21)
\end{aligned}$$

e. Assume that the density n and relative velocities V are constant over time, we have

$$\int_0^{t_{\mathrm{relax}}} \langle dV^2 \rangle \mathrm{d}t = \frac{8\pi G^2 m^2 n}{V} \ln \Lambda t_{\mathrm{relax}} = V^2 \tag{6.2.22}$$

i.e.

$$t_{\mathrm{relax}} = \frac{V^3}{8\pi G^2 m^2 n \ln \Lambda} \tag{6.2.23}$$

Solutions, Cosmology 2016/2017, Week 7

7.1 Rotation of "Spiral Nebulae"

1. Rotation period = $2\pi\ 5 \times 60''/(0.022''\mathrm{yr}^{-1}) \approx 86000$ yr
2. Speed = $2\pi\times$ 15 kpc / 86000 yr $\approx$ 1.1 pc/yr = 1.07×10^9 m/s = $3.6\,c$!
3. Proper motion on the sky = $0.022''$ yr^{-1}. Absolute velocity = 100 km/s = 1.0×10^{-4} pc yr^{-1}.
   Distance where $0.022''$ corresponds to 10^{-4} pc: 10^{-4} pc / $D = \tan 0.022''$ i.e. $D \approx 960$ pc. Well within Shapley's Galaxy.
4. Plate scale = $30''$ mm^{-1}. Shift = $15 \times 0.022'' = 0.33'' \approx 0.01$ mm

7.2 Radial velocities and radiation pressure

As per assumption (1), masses are

$$M = \frac{r_p v^2}{G} = \frac{r_a D v^2}{G} \tag{7.2.24}$$

where r_p is the physical radius, r_a is the angular radius, D the distance and v the rotational velocity.

According to assumption (4), the radiation pressure is distance-independent. The momentum of a photon is

$$p = E/c \tag{7.2.25}$$

for photon energy E. Hence, the radiation pressure from a star of magnitude 1 is

$$\mathcal{P} = \frac{L_\odot}{4\pi(1\mathrm{AU})^2}\frac{1}{1.2\times 10^{11}c} \tag{7.2.26}$$

and the radiation pressure from one square degree

$$\mathcal{P} = 0.035 \times \frac{L_\odot}{4\pi(1\mathrm{AU})^2}\frac{1}{1.2\times 10^{11}c} \tag{7.2.27}$$

$$= 2.92\times 10^{-13}\frac{L_\odot}{4\pi c(1\mathrm{AU})^2} \tag{7.2.28}$$

There are $2\pi(180/\pi)^2 = 20626$ square degrees in a hemisphere. If all the light came from one point, along a line-of-sight perpendicular to the plane of a nebula, the pressure would then be

$$\mathcal{P} = 6.0\times 10^{-9}\frac{L_\odot}{4\pi c(1\mathrm{AU})^2} \tag{7.2.29}$$

Since it is distributed over a hemisphere, the actual pressure is half that,

$$\mathcal{P} = 3.0\times 10^{-9}\frac{L_\odot}{4\pi c(1\mathrm{AU})^2} \tag{7.2.30}$$

Hence, the force on the nebula is

$$F = \mathcal{P}\times\pi R_p^2 \tag{7.2.31}$$

$$= 3.0\times 10^{-9}\frac{D^2 R_a^2 L_\odot}{4c(1\mathrm{AU})^2} \tag{7.2.32}$$

where R_p and R_a are the "outer" physical and angular radii. Hence, the acceleration is

$$A = F/M = 3.0 \times 10^{-9} \frac{D^2 R_a^2 L_\odot}{4c(1\mathrm{AU})^2} \frac{G}{r_a D v^2} \quad (7.2.33)$$

$$= 7.5 \times 10^{-10} \frac{L_\odot G}{c(1\mathrm{AU})^2} \frac{D R_a^2}{r_a v^2} \quad (7.2.34)$$

- For $D = 4.4 \times 10^{22}$ m, $r = 150'' = 7.27 \times 10^{-4}$ rad, $R = 210'' = 1.02 \times 10^{-3}$ rad, we get $A = 1.04 \times 10^{-15}$ m/s^2.
- Time to accelerate to 1000 km/s = 3×10^{13} years.
- Distance travelled:

$$D = \int_0^t v(\tau) \mathrm{d}\tau = \int_0^t A\, \tau \mathrm{d}\tau = \frac{1}{2} A\, t^2$$

  Inserting $t = 9.62 \times 10^{20}$ s and $A = 1.04 \times 10^{-15}$ m/s^2, we find $D = 4.8 \times 10^{26}$ m = 1.6×10^{10} pc. Leads to many evident inconsistencies.

The most obvious effect that has been ignored is, of course, gravity from the Milky Way. This would counteract the acceleration from radiation pressure, although the exact gravitation of the Milky Way was difficult to estimate in 1921 as the mass of the Milky Way was very poorly constrained.

Also, it is clearly not realistic to assume that the Milky Way occupies half the sky, as seen from a distant galaxy. This would further reduce the radiation pressure.

7.3 Hot gas in galaxy clusters

Velocity of particles in gas with temperature T:

$$v_{\mathrm{rms}} = \sqrt{\frac{3kT}{\mu}} \quad (7.3.1)$$

i.e.

$$T = \frac{v_{\mathrm{rms}}^2 \mu}{3k} = \frac{\sigma^2 \mu}{k} \quad (7.3.2)$$

v_{rms} is the 3-D velocity dispersion, i.e. $v_{\mathrm{rms}} = 3\sigma_{1\mathrm{D}}$ and μ is the mean molecular weight. Inserting $\mu = 10^{-27}$ kg and $v_{\mathrm{rms}} = 10^6$ m s^{-1} yields $T = 72 \times 10^6$ K and it is clear from the above that T scales with σ^2, i.e.

$$T = 72 \times 10^6 \left(\frac{\sigma_{1\mathrm{D}}}{1000\,\mathrm{km\,s^{-1}}} \right)^2 \mathrm{K} \quad (7.3.3)$$

Solutions, Cosmology 2016/2017, Week 12

12.1 Cosmological surface brightness dimming

The flux is

$$F(z) = \frac{L}{4\pi D_L^2} = \frac{L}{4\pi D^2(1+z)^2} \tag{12.1.1}$$

for distance measure D, redshift z and luminosity L. The angular size is

$$\theta(z) \propto \frac{1}{D_A} \propto \frac{(z+1)}{D} \tag{12.1.2}$$

and the solid angle is then

$$\Omega(z) \propto \frac{(1+z)^2}{D^2} \tag{12.1.3}$$

We thus find the intensity scaling as

$$I(z) = \frac{F(z)}{\Omega(z)} \propto \frac{L}{4\pi D^2(1+z)^2}\frac{D^2}{(1+z)^2} \propto \frac{1}{(1+z)^4} \tag{12.1.4}$$

12.2 Cosmological distances

a. In general, we have

$$r = \int_{t_0}^{t_1} \frac{c}{a(t)}\mathrm{d}t \tag{12.2.1}$$

and for Einstein-de Sitter:

$$a(t) = \left(\frac{3H_0 t}{2}\right)^{2/3} \tag{12.2.2}$$

Converting to an integral over redshift:

$$a(t) = (1+z)^{-1} \tag{12.2.3}$$

i.e.

$$z = 1/a(t) - 1 \tag{12.2.4}$$

so

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}a}\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{1}{a(t)^2}\dot{a}(t) \tag{12.2.5}$$

Let's try to express $\dot{a}$ in terms of z:

$$\dot{a} = \frac{2}{3}\left(\frac{3H_0}{2}\right)^{2/3} t^{-1/3} \tag{12.2.6}$$

$$\dot{a}^2 = \frac{4}{9}\left(\frac{3H_0}{2}\right)^{4/3} t^{-2/3} \tag{12.2.7}$$

We have

$$a(t) = \left(\frac{3H_0}{2}\right)^{2/3} t^{2/3} \tag{12.2.8}$$

so

$$t^{-2/3} = \left(\frac{3H_0}{2}\right)^{2/3} a(t)^{-1} \tag{12.2.9}$$

i.e.

$$\dot{a}^2 = \frac{4}{9}\left(\frac{3H_0}{2}\right)^{4/3}\left(\frac{3H_0}{2}\right)^{2/3} a(t)^{-1} \tag{12.2.10}$$

$$= \frac{4}{9}\left(\frac{3H_0}{2}\right)^{4/3}\left(\frac{3H_0}{2}\right)^{2/3} (1+z) \tag{12.2.11}$$

$$= \frac{4}{9}\left(\frac{3H_0}{2}\right)^{2} (1+z) \tag{12.2.12}$$

$$= H_0^2(1+z) \tag{12.2.13}$$

So now we can integrate over z:

$$r = -\int_0^z \frac{c}{a(t)}\frac{a(t)^2}{\dot{a}(t)}\mathrm{d}z = -c\int_0^z \frac{a(t)}{\dot{a}(t)}\mathrm{d}z \tag{12.2.14}$$

$$= -c\int_0^z \frac{(1+z)^{-1}}{H_0(1+z)^{1/2}}\mathrm{d}z \tag{12.2.15}$$

$$= -\frac{c}{H_0}\int_0^z (1+z)^{-3/2}\mathrm{d}z \tag{12.2.16}$$

$$= -\frac{c}{H_0}\left(2 - \frac{2}{\sqrt{1+z}}\right) \tag{12.2.17}$$

$$= -2\frac{c}{H_0}\left(1-(1+z)^{-1/2}\right) \tag{12.2.18}$$

b.

$$D_A = D/(1+z) = r/(1+z) \tag{12.2.19}$$

$$= 2\frac{c}{H_0}\left(1-(1+z)^{-1/2}\right)/(1+z) \tag{12.2.20}$$

$$= 2\frac{c}{H_0}\left((1+z)^{-1}-(1+z)^{-3/2}\right) \tag{12.2.21}$$

This has extremum for

$$\frac{\mathrm{d}D_A}{\mathrm{d}z} = 0 \tag{12.2.22}$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}z}(1+z)^{-1} = \frac{\mathrm{d}}{\mathrm{d}z}(1+z)^{-3/2} \tag{12.2.23}$$

$$-(1+z)^{-2} = -\frac{3}{2}(1+z)^{-5/2} \tag{12.2.24}$$

$$\frac{2}{3} = (1+z)^{-1/2} \tag{12.2.25}$$

$$\frac{9}{4} = (1+z) \tag{12.2.26}$$

$$\frac{9}{4} - 1 = \frac{5}{4} = z \tag{12.2.27}$$

12.3 Sunyaev-Zeldovich effect

Rayleigh-Jeans approximation:

$$I_\nu = \frac{2\nu^2 kT}{c^2} \tag{12.3.1}$$

The energy of each photon increases by $\Delta E_\nu / E_\nu = y$, equivalently $\Delta\nu/\nu = y$. We first convince ourselves that a constant fractional increase in the frequency of each photon simply shifts the spectral energy distribution horizontally:

Consider n photons per unit time in the frequency interval $\delta\nu_0$ at frequency ν_0. The scattering process does not change the number of photons. Before shifting, the photons carry an energy per unit frequency interval of

$$\delta E/\delta\nu = \delta E_0/\delta\nu_0 = nh\nu_0/\delta\nu_0 \tag{12.3.2}$$

After shifting by factor $(1+y)$, the interval is mapped onto

$$\delta\nu' = (1+y)\delta\nu_0 \tag{12.3.3}$$

at frequency $\nu' = (1+y)\nu_0$, and the energy is $\delta E' = nh\nu'$. The energy per unit frequency is now

$$\delta E'/\delta\nu' = \delta E'/\delta\nu' \tag{12.3.4}$$

$$= nh\nu'/\delta\nu' \tag{12.3.5}$$

$$= \frac{nh(1+y)\nu_0}{(1+y)\delta\nu_0} \tag{12.3.6}$$

$$= \delta E_0/\delta\nu_0 \tag{12.3.7}$$

Hence, the energy per unit frequency (per unit time, i.e. the specific intensity, I_ν) remains unchanged for a fixed point on the spectral energy distribution - the curve simply shifts horizontally in frequency space.

Now, for brevity define $\alpha = 2kT/c^2$:

$$I_\nu = \alpha\nu^2 \tag{12.3.8}$$

The slope of the curve is

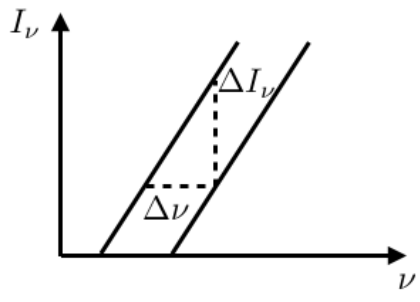
$$\mathrm{d}I_\nu/\mathrm{d}\nu = 2\alpha\nu \tag{12.3.9}$$

If the curve is shifted by $\Delta\nu$, the change in intensity at fixed ν is then (see sketch below)

$$\Delta I_\nu = -(\mathrm{d}I_\nu/\mathrm{d}\nu)\Delta\nu = -2\alpha\nu\Delta\nu \tag{12.3.10}$$

Dividing through by I_ν:

$$\Delta I_\nu/I_\nu = -2\alpha\nu\Delta\nu/(\alpha\nu^2) = -2\Delta\nu/\nu = -2y \tag{12.3.11}$$



Solutions, Cosmology 2016/2017, Week 13

13.1 Gravitational microlensing

- We require

$$\theta_E > \theta_{\rm src} \tag{13.1.1}$$

where $\theta_{\rm src}$ is the radius of the source.

$$\frac{4GM_L}{c^2}\left(\frac{D_{\rm LS}}{D_{\rm S}D_{\rm L}}\right) > (R_{\rm src}/D_{\rm S})^2 \tag{13.1.2}$$

$$M_L > \frac{R_{\rm src}^2 c^2}{4G}\left(\frac{D_{\rm L}}{D_{\rm LS}D_{\rm S}}\right) \tag{13.1.3}$$

For $R_{\rm src} = R_\odot = 7\times10^8$ m, $D_{\rm L} = 10$ kpc, $D_{\rm LS} = 40$ kpc, $D_{\rm S} = 50$ kpc we get $M_L = 2.65\times10^{22}$ kg $= 1.3\times10^{-8}M_\odot$ (see also Paczyński 1986).

- For a shell of density ρ and thickness dr, the mass surface density is

$$\mathrm{d}\sigma_M = \rho \mathrm{d}r \tag{13.1.4}$$

and the surface density of lenses (per unit area) is

$$\mathrm{d}\sigma_L = \frac{\rho}{M_L}\mathrm{d}r \tag{13.1.5}$$

For shell radius r, the surface density of lenses per unit solid angle is

$$\mathrm{d}\Sigma_L = r^2 \mathrm{d}\sigma_L \tag{13.1.6}$$

The optical depth is

$$\mathrm{d}\tau = \pi \mathrm{d}\Sigma_L \theta_E^2 \tag{13.1.7}$$

where

$$\theta_E^2 = \frac{4GM_L}{c^2}\left(\frac{D_{\rm S}-r}{D_{\rm S}r}\right) \tag{13.1.8}$$

The total optical depth for a system of radius R is then

$$\begin{aligned}
\tau &= \int_0^R \frac{4GM_L}{c^2}\left(\frac{D_{\rm S}-r}{D_{\rm S}r}\right)\frac{\pi r^2\rho}{M_L}\mathrm{d}r \\
&= \frac{4\pi G\rho}{c^2}\int_0^R \left(\frac{D_{\rm S}-r}{D_{\rm S}}\right) r\,\mathrm{d}r \\
&= \frac{4\pi G\rho}{c^2}\int_0^R (r - r^2/D_S)\mathrm{d}r \\
&= \frac{4\pi G\rho}{c^2}\left(\frac{R^2}{2} - \frac{R^3}{3D_S}\right)
\end{aligned}$$

For $D_S = R$, we then have

$$\tau = \left(\frac{2\pi}{3}\right)\left(\frac{G\rho}{c^2}\right)D_S^2 \tag{13.1.9}$$

(Paczyński 1996, Eq. 20)

- For a uniform sphere of radius D_S, we have from the virial theorem that

$$M = 5\frac{\sigma^2 D_S}{G} \tag{13.1.10}$$

that is

$$\frac{4}{3}\pi\rho D_S^3 = 5\frac{\sigma^2 D_S}{G} \tag{13.1.11}$$

or

$$D_S^2 = \frac{15}{4}\frac{\sigma^2}{\pi\rho G} \tag{13.1.12}$$

Using the expression for the optical depth, τ, derived before:

$$\begin{aligned} \tau &= \left(\frac{2\pi}{3}\right)\left(\frac{G\rho}{c^2}\right)\frac{15}{4}\frac{\sigma^2}{\pi\rho G} \\ &= \left(\frac{5}{2}\right)\left(\frac{\sigma^2}{c^2}\right) \end{aligned}$$

(Paczyński 1996, Eq. 22)

13.2 The flatness problem

The critical density is defined at any epoch as

$$\rho_c = 3H^2/8\pi G = 3(\dot{a}/a)^2/8\pi G \tag{13.2.1}$$

so the density parameter Ω_M is

$$\Omega_M = \rho_M/\rho_c = \frac{8\pi G\rho_M}{3(\dot{a}/a)^2} \tag{13.2.2}$$

We have

$$\dot{a} = H_0\left[\Omega_0(1/a-1) + \Omega_\Lambda(a^2-1) + 1\right]^{1/2} \tag{13.2.3}$$

and

$$\rho_M = \rho_0 a^{-3} = (3\Omega_0 H_0^2/8\pi G)a^{-3} \tag{13.2.4}$$

so

$$\Omega_M = \frac{8\pi G(3\Omega_0 H_0^2/8\pi G)a^{-3}}{3H_0^2\left[\Omega_0(1/a-1) + \Omega_\Lambda(a^2-1) + 1\right]/a^2} \tag{13.2.5}$$

$$= \frac{\Omega_0 a^{-1}}{\left[\Omega_0(1/a-1) + \Omega_\Lambda(a^2-1) + 1\right]} \tag{13.2.6}$$

$$= \frac{\Omega_0}{\Omega_0(1-a) + \Omega_\Lambda(a^3-a) + a} \tag{13.2.7}$$

(13.2.8)

We then see that for small a this reduces to

$$\lim_{a\to 0}\Omega_M = 1 \tag{13.2.9}$$

or, equivalently,

$$\lim_{z\to\infty}\Omega_M = 1 \tag{13.2.10}$$

At redshift $z = 1000$ we have $a = 10^{-3}$ so that $1 - \Omega_M \approx 3\times 10^{-9}$.

13.3 Parametric solutions to Friedman's equation

In an earlier lecture we found that the Friedman equation can be written as

$$\dot{a} = H_0 \left[\Omega_0(1/a - 1) + \Omega_\Lambda(a^2 - 1) + 1\right]^{1/2} \tag{13.3.1}$$

By assumption, we here have $\Omega_\Lambda = 0$ so

$$\dot{a} = H_0 \left[\Omega_0(1/a - 1) + 1\right]^{1/2} \tag{13.3.2}$$

We have to show that the parametric solutions indeed satisfy this relation. They are:

$$a(\theta) = \frac{\Omega_0}{2(\Omega_0 - 1)}(1 - \cos\theta) \tag{13.3.3}$$

$$t(\theta) = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}(\theta - \sin\theta) \tag{13.3.4}$$

Hence, we have

$$\mathrm{d}a = \frac{\mathrm{d}a}{\mathrm{d}\theta}\mathrm{d}\theta = \frac{\Omega_0}{2(\Omega_0 - 1)}\sin\theta\,\mathrm{d}\theta \tag{13.3.5}$$

$$\mathrm{d}t = \frac{\mathrm{d}t}{\mathrm{d}\theta}\mathrm{d}\theta = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}(1 - \cos\theta)\,\mathrm{d}\theta \tag{13.3.6}$$

We then find

$$\dot{a} = \frac{\frac{\Omega_0}{2(\Omega_0-1)}\sin\theta}{\frac{\Omega_0}{2H_0(\Omega_0-1)^{3/2}}(1 - \cos\theta)} = H_0(\Omega_0 - 1)^{1/2}\frac{\sin\theta}{1 - \cos\theta} \tag{13.3.7}$$

Squaring Eq. (13.3.2), inserting Eq. (13.3.7) on the left-hand side and Eq. (13.3.3) on the right-hand side, we find that the solution should satisfy

$$\frac{\sin^2\theta}{(1 - \cos\theta)^2} = \frac{2}{(1 - \cos\theta)} - 1 \tag{13.3.8}$$

If we multiply by $(1 - \cos\theta)^2$, we find

$$\sin^2\theta = 2(1 - \cos\theta) - (1 - \cos\theta)^2 \tag{13.3.9}$$

$$= 2 - 2\cos\theta - 1 - \cos^2\theta + 2\cos\theta \tag{13.3.10}$$

$$= 1 - \cos^2\theta \tag{13.3.11}$$

which is, of course a valid trigonometric identity. Hence, the solutions (13.3.3) and (13.3.4) do indeed satisfy (13.3.2).

13.4 Tophat model

Virial equilibrium is reached once the density contrast has re-collapsed back to half its maximum size, $a_{\rm vir} = \frac{1}{2}a_{\rm max}$. This means that $\cos\theta = 0$ in Eq. (13.3.3), corresponding to $\theta = \pi/2, 3\pi/2, \ldots$. The relevant solution here is $\theta_{\rm vir} = 3\pi/2$. At this point the background scale factor is

$$a = \left(\frac{3H_0 t_{\rm vir}}{2}\right)^{2/3} \tag{13.4.1}$$

while

$$a_{\rm vir} = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1} \tag{13.4.2}$$

The density contrast is then

$$\begin{aligned}
\Delta_{\rm vir} &= (a/a_{\rm vir})^3 && (13.4.3)\\
&= \left(\frac{3H_0 t_{\rm vir}}{2}\right)^2 \left(\frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}\right)^{-3} && (13.4.4)\\
&= \left(\frac{3H_0 \frac{\Omega_0}{2H_0(\Omega_0-1)^{3/2}}(\theta_{\rm vir} - \sin\theta_{\rm vir})}{2}\right)^2 \left(\frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}\right)^{-3} && (13.4.5)\\
&= \frac{9}{2}\Omega_0(\theta_{\rm vir} - \sin\theta_{\rm vir})^2 \approx 150 && (13.4.6)\\
& && (13.4.7)
\end{aligned}$$

since $\Omega_0 = 1$.

13.5 The Press-Schechter mass function

The derivation closely follows that given in the lecture slides for the case $\Sigma^2 = \sigma^2 V$. This in turns follows the original paper by Press & Schechter (1974; PS74), which may be consulted for more in-depth discussion. Here we suppose that

$$\Sigma^2 = \sigma^2 V^{2\alpha} \tag{13.5.1}$$

The *relative* fluctuations per volume are then

$$\Sigma(V)/V = \sqrt{V^{2\alpha}\sigma^2}/V = \sigma V^{\alpha-1} \tag{13.5.2}$$

or, per mass (for mean density ρ, so that $M(V) = \rho V$):

$$\Sigma(V)/M(V) = \frac{\sqrt{V^{2\alpha}\sigma^2}}{\rho V} = \frac{\sigma}{\rho}V^{\alpha-1} \tag{13.5.3}$$

For fractional difference between the mean mass $\langle M(V)\rangle$ and actual mass in a particular volume $M(V)$,

$$\Delta \equiv \frac{M(V) - \langle M(V)\rangle}{\langle M(V)\rangle} \tag{13.5.4}$$

the probability density function $p(\delta, V)$ is then a Gaussian with mean 0 and dispersion

$$\Delta_\star = \Sigma(V)/M(V) = \frac{\sigma}{\rho}V^{\alpha-1} \tag{13.5.5}$$

As before, the probability that a volume V is bound by a_2 is

$$P = \frac{1}{2}\,{\rm erfc}\left(\frac{\Delta_{\rm crit}a_1}{\sqrt{2}\Delta_\star a_2}\right) \tag{13.5.6}$$

but we now substitute the expression (13.5.5) so that

$$\begin{aligned}
P &= \frac{1}{2}\,{\rm erfc}\left(\frac{\rho}{\sigma}\frac{\Delta_{\rm crit}a_1}{\sqrt{2}a_2}V^{1-\alpha}\right) && (13.5.7)\\
&= \frac{1}{2}\,{\rm erfc}\left(\frac{1}{\sigma}\frac{\Delta_{\rm crit}a_1}{\sqrt{2}a_2}M^{1-\alpha}\rho^{\alpha}\right) && (13.5.8)
\end{aligned}$$

which is Eq. 18 in PS74. In general,

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{erfc}\left(a\xi^{b}\right) = -\frac{2ab\exp\left(-a^{2}\xi^{2b}\right)\xi^{b-1}}{\sqrt{\pi}} \tag{13.5.9}$$

The differential probability distribution is then

$$\frac{\mathrm{d}P}{\mathrm{d}M} = \frac{1}{2}\frac{2}{\sigma\sqrt{\pi}}\frac{\Delta_{\mathrm{crit}}a_{1}}{\sqrt{2}a_{2}}\rho^{\alpha}(1-\alpha)M^{-\alpha}\exp\left(-\frac{1}{\sigma^{2}}\frac{\Delta_{\mathrm{crit}}^{2}a_{1}^{2}}{2a_{2}^{2}}\rho^{2\alpha}M^{2-2\alpha}\right) \tag{13.5.10}$$

and the number density is

$$\frac{\mathrm{d}N}{\mathrm{d}M} = \rho_{1}^{1+\alpha}\left(\frac{a_{1}}{a_{2}}\right)^{4}\frac{\Delta_{\mathrm{crit}}}{\sigma}\sqrt{2/\pi}(1-\alpha)M^{-\alpha-1}\exp\left(-\frac{1}{\sigma^{2}}\frac{\Delta_{\mathrm{crit}}^{2}a_{1}^{2}}{2a_{2}^{2}}\rho^{2\alpha}M^{2(1-\alpha)}\right) \tag{13.5.11}$$

which is Eq. (19) in PS74, apart from the factor of two that PS74 introduce to account for the underdensities. Eq. 13.5.11 is thus of the form

$$\frac{\mathrm{d}N}{\mathrm{d}M} \propto M^{-1-\alpha}\exp\left(-\left[\frac{M}{M^{\star}}\right]^{2(1-\alpha)}\right) \tag{13.5.12}$$

where $M^{\star} \propto a_{2}^{2/[2(1-\alpha)]} = a_{2}^{1/(1-\alpha)}$

Solutions, Cosmology 2016/2017, Week 14

14.1 Decaying potentials

Assume for simplicity that the perturbation is spherical. For a particle located at physical distance R from the center, the Newtonian potential is then (from the shell theorem)

$$\Psi \equiv -\frac{GM}{R} \tag{14.1.1}$$

hence the part of this due to a perturbation of mass δM is

$$\delta\Psi = -\frac{G\delta M}{R} \tag{14.1.2}$$

In an expanding Universe, $R \propto a$, so if the overdensities cannot grow then we immediately see that

$$\delta\Psi \propto a^{-1} \tag{14.1.3}$$

i.e., the perturbations of the potential $\delta\Psi$ decay as the Universe expands, as long as the perturbations grow more slowly than a.

In the linear regime, we have

$$\frac{\delta\rho}{\rho} \propto a \tag{14.1.4}$$

For a fixed *co-moving* distance r from the centre, $M(r) \approx$ const, while the physical distance is $R = ar$. So we have

$$\frac{\delta M}{M} = \frac{\delta\rho}{\rho} \propto a \tag{14.1.5}$$

so

$$\delta\Psi = -\frac{G\delta M}{R} \propto -\frac{Ga}{ar} = \text{const} \tag{14.1.6}$$

14.2 Newtonian equivalence of metric perturbations

- The general expression for the Christoffel symbol is

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{14.2.1}$$

For $\Gamma^{i}{}_{00}$ we have

$$\Gamma^{i}{}_{00} = \frac{g^{i\nu}}{2}\left[\frac{\partial g_{0\nu}}{\partial x^{0}} + \frac{\partial g_{0\nu}}{\partial x^{0}} - \frac{\partial g_{00}}{\partial x^{\nu}}\right] \tag{14.2.2}$$

Since the metric is diagonal, this is non-zero only for $\nu = i$ so we have

$$\begin{aligned}
\Gamma^{i}{}_{00} &= \frac{g^{ii}}{2}\left[\frac{\partial g_{0i}}{\partial x^{0}} + \frac{\partial g_{0i}}{\partial x^{0}} - \frac{\partial g_{00}}{\partial x^{i}}\right] && (14.2.3)\\
&= \frac{1+2\phi}{2}\left[0 + 0 - \frac{\partial(-1-2\phi)}{\partial x^{i}}\right] && (14.2.4)\\
&= \frac{1+2\phi}{2}\left[2\frac{\partial\phi}{\partial x^{i}}\right] && (14.2.5)\\
&= (1+2\phi)\frac{\partial\phi}{\partial x^{i}} && (14.2.6)\\
&\simeq \frac{\partial\phi}{\partial x^{i}} && (14.2.7)
\end{aligned}$$

where the last step follows by elimination of the second-order term $\phi \frac{\partial\phi}{\partial x^i}$.

- In addition to $\Gamma^i{}_{00}$ from above, we also need the other spatial Christoffel symbols:

$$\Gamma^k{}_{ij} = \frac{g^{k\nu}}{2}\left[\frac{\partial g_{i\nu}}{\partial x^j} + \frac{\partial g_{j\nu}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\nu}\right] \tag{14.2.8}$$

$$= \frac{g^{kk}}{2}\left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right] \tag{14.2.9}$$

These are non-zero only for $i = j = k$, in which case we get

$$\Gamma^i{}_{ii} = \frac{g^{ii}}{2}\left[\frac{\partial g_{ii}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^i}\right] \tag{14.2.10}$$

$$= \frac{g^{ii}}{2}\left[\frac{\partial g_{ii}}{\partial x^i}\right] \tag{14.2.11}$$

$$= \frac{1-2\phi}{2}\left[\frac{\partial(1-2\phi)}{\partial x^i}\right] \tag{14.2.12}$$

$$= -\frac{\partial\phi}{\partial x^i} \tag{14.2.13}$$

And

$$\Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \frac{g^{ii}}{2}\left[\frac{\partial g_{0i}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^0} - \frac{\partial g_{0j}}{\partial x^i}\right] \tag{14.2.14}$$

$$= \frac{g^{ii}}{2}\left[\frac{\partial g_{ji}}{\partial x^0}\right] \tag{14.2.15}$$

Again, non-zero only for $i = j$, where we get

$$\Gamma^i{}_{0i} = \frac{1-2\phi}{2}\left[\frac{\partial 1 - 2\phi}{\partial t}\right] \tag{14.2.16}$$

$$= -\frac{\partial\phi}{\partial t} \tag{14.2.17}$$

Then we can go on to look at the geodesic equation:

$$\frac{d^2x^i}{d\lambda^2} = -\left[\Gamma^i{}_{00}\frac{dx^0}{d\lambda}\frac{dx^0}{d\lambda} + \Gamma^i{}_{0i}\frac{dx^0}{d\lambda}\frac{dx^i}{d\lambda} + \Gamma^i{}_{i0}\frac{dx^i}{d\lambda}\frac{dx^0}{d\lambda} + \Gamma^i{}_{ii}\frac{dx^i}{d\lambda}\frac{dx^i}{d\lambda}\right] \tag{14.2.18}$$

$$= -\left[\frac{\partial\phi}{\partial x^i}\frac{dx^0}{d\lambda}\frac{dx^0}{d\lambda} - \frac{\partial\phi}{\partial t}\frac{dx^0}{d\lambda}\frac{dx^i}{d\lambda} - \frac{\partial\phi}{\partial t}\frac{dx^i}{d\lambda}\frac{dx^0}{d\lambda} - \frac{\partial\phi}{\partial x^i}\frac{dx^i}{d\lambda}\frac{dx^i}{d\lambda}\right] \tag{14.2.19}$$

$$= -\left[\frac{\partial\phi}{\partial x^i}P^0P^0 - \frac{\partial\phi}{\partial t}P^0P^i - \frac{\partial\phi}{\partial t}P^0P^i - \frac{\partial\phi}{\partial x^i}P^iP^i\right] \tag{14.2.20}$$

$$= \frac{\partial\phi}{\partial x^i}\left[(P^i)^2 - (P^0)^2\right] \tag{14.2.21}$$

$$\simeq -\frac{\partial\phi}{\partial x^i}(P^0)^2 \tag{14.2.22}$$

$$\tag{14.2.23}$$

On the left, we have

$$\frac{d^2x^i}{d\lambda^2} = \frac{d}{d\lambda}\frac{dx^i}{d\lambda} = \frac{dP^i}{d\lambda} = \frac{dP^i}{dt}\frac{dt}{d\lambda} = \frac{dP^i}{dt}P^0 \tag{14.2.24}$$

Then we get

$$\frac{\mathrm{d}P^i}{\mathrm{d}t}P^0 = -\frac{\partial\phi}{\partial x^i}(P^0)^2 \tag{14.2.25}$$

$$\frac{\mathrm{d}P^i}{\mathrm{d}t} = -\frac{\partial\phi}{\partial x^i}P^0 \tag{14.2.26}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}x_i}{\mathrm{d}t}\right) = m\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = -\frac{\partial\phi}{\partial x^i}P^0 \tag{14.2.27}$$

For a non-relativistic particle we have $P^0 = E = m(c^2) + \frac{1}{2}mv^2$, which is dominated by the rest mass term, $E \simeq m(c^2)$. Thus m cancels out and we get

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = -\frac{\partial\phi}{\partial x^i} \tag{14.2.28}$$

which is what we wanted to show.

14.3 Four-momentum of photons in perturbed FRW metric

For the other P^i: we expand the g_{ij} and get

$$p^2 = g_{ij}P^iP^j \tag{14.3.1}$$

$$p^i = \left[a^2(1+2\Phi)\right]^{1/2}(P^i) \tag{14.3.2}$$

$$P^i = p^i\left[a^2(1+2\Phi)\right]^{-1/2} \tag{14.3.3}$$

$$\approx \frac{p^i}{a}(1-\Phi) \tag{14.3.4}$$

$$= p\hat{p}^i\frac{1-\Phi}{a} \tag{14.3.5}$$

14.4 The momentum time derivative

- We start from the geodesic equation,

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} = -\Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} \tag{14.4.1}$$

and use the definition of the four-momentum,

$$P^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{14.4.2}$$

The zeroth component of the geodesic eqn. is then

$$\frac{\mathrm{d}P^0}{\mathrm{d}\lambda} = -\Gamma^0{}_{\alpha\beta}P^\alpha P^\beta \tag{14.4.3}$$

For the left-hand side, we have

$$\frac{\mathrm{d}P^0}{\mathrm{d}\lambda} = \frac{\mathrm{d}P^0}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{\mathrm{d}P^0}{\mathrm{d}t}P^0 \tag{14.4.4}$$

$$= P^0\frac{\mathrm{d}}{\mathrm{d}t}p(1-\Psi) \tag{14.4.5}$$

so that

$$P^0 \frac{\mathrm{d}}{\mathrm{d}t} p(1-\Psi) = -\Gamma^0{}_{\alpha\beta} P^\alpha P^\beta \tag{14.4.6}$$

or

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} p(1-\Psi) &= -\Gamma^0{}_{\alpha\beta} \frac{P^\alpha P^\beta}{P^0} && (14.4.7)\\
&= -\Gamma^0{}_{\alpha\beta} \frac{P^\alpha P^\beta}{p(1-\Psi)} && (14.4.8)\\
&= -\Gamma^0{}_{\alpha\beta} \frac{P^\alpha P^\beta}{p}(1+\Psi) && (14.4.9)
\end{aligned}$$

which is Eq. (4.23), p. 91.

- Next, the left-hand side is expanded:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} p(1-\Psi) &= \frac{\mathrm{d}}{\mathrm{d}t} p - p\Psi && (14.4.10)\\
&= \frac{\mathrm{d}p}{\mathrm{d}t} - \left(\frac{\mathrm{d}p}{\mathrm{d}t}\Psi + p\frac{\mathrm{d}\Psi}{\mathrm{d}t}\right) && (14.4.11)\\
&= \frac{\mathrm{d}p}{\mathrm{d}t}(1-\Psi) - p\frac{\mathrm{d}\Psi}{\mathrm{d}t} && (14.4.12)
\end{aligned}$$

Inserting this into Eq. (14.4.9), we get

$$\frac{\mathrm{d}p}{\mathrm{d}t}(1-\Psi) - p\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -\Gamma^0{}_{\alpha\beta} \frac{P^\alpha P^\beta}{p}(1+\Psi) \tag{14.4.13}$$

or

$$\frac{\mathrm{d}p}{\mathrm{d}t}(1-\Psi) = p\frac{\mathrm{d}\Psi}{\mathrm{d}t} - \Gamma^0{}_{\alpha\beta} \frac{P^\alpha P^\beta}{p}(1+\Psi) \tag{14.4.14}$$

which is Eq. (4.24).

- Next, multiply both sides by $(1+\Psi)$ and continue dropping terms that are quadratic in Ψ:

$$\frac{\mathrm{d}p}{\mathrm{d}t}(1-\Psi)(1+\Psi) = p\frac{\mathrm{d}\Psi}{\mathrm{d}t}(1+\Psi) - \Gamma^0{}_{\alpha\beta} \frac{P^\alpha P^\beta}{p}(1+\Psi)^2 \tag{14.4.15}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t}(1-\Psi^2) = p\frac{\mathrm{d}\Psi}{\mathrm{d}t}(1+\Psi) - \Gamma^0{}_{\alpha\beta} \frac{P^\alpha P^\beta}{p}(1+2\Psi) \tag{14.4.16}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p\frac{\mathrm{d}\Psi}{\mathrm{d}t}(1+\Psi) - \Gamma^0{}_{\alpha\beta} \frac{P^\alpha P^\beta}{p}(1+2\Psi) \tag{14.4.17}$$

Express $\mathrm{d}\Psi/\mathrm{d}t$ in terms of partial derivatives:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \frac{\partial\Psi}{\partial t} + \frac{\partial\Psi}{\partial x^i}\frac{\partial x^i}{\partial t} \tag{14.4.18}$$

where

$$\frac{\partial x^i}{\partial t} = \frac{\hat{p}^i}{a}(1+\Psi-\Phi) \tag{14.4.19}$$

so

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \frac{\partial\Psi}{\partial t} + \frac{\partial\Psi}{\partial x^i}\frac{\hat{p}^i}{a}(1+\Psi-\Phi) \tag{14.4.20}$$

Inserting this in Eq. (14.4.17), we get

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p\left(\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}(1+\Psi-\Phi)\right)(1+\Psi) - \Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+2\Psi) \tag{14.4.21}$$

There are some more second-order terms to be gotten rid off:

$$\begin{aligned}
&\left(\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}(1+\Psi-\Phi)\right)(1+\Psi) && (14.4.22)\\
=\; &\Psi\left(\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}(1+\Psi-\Phi)\right) + \left(\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}(1+\Psi-\Phi)\right) && (14.4.23)\\
=\; &\Psi\frac{\partial\Psi}{\partial t} + \Psi\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}(1+\Psi-\Phi) + \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}(1+\Psi-\Phi) && (14.4.24)
\end{aligned}$$

Removing terms involving Ψ^2 or $\Psi\Phi$ leaves:

$$= \Psi\frac{\partial\Psi}{\partial t} + \Psi\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} + \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}(1+\Psi-\Phi) \tag{14.4.25}$$

But also the terms $\partial\Psi/\partial x^i$ and $\partial\Psi/\partial t$ are first-order terms (i.e., non-zero only for perturbed solutions) so we can also remove terms involving $\Psi\partial\Psi/\partial x^i$, $\Phi\partial\Psi/\partial x^i$, and $\Psi\partial\Psi/\partial t$, which are then second-order:

$$= \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} \tag{14.4.26}$$

We have now reduced Eq. (14.4.21) to

$$\boxed{\frac{\mathrm{d}p}{\mathrm{d}t} = p\left(\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right) - \Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+2\Psi)} \tag{14.4.27}$$

which is Eq. (4.25).

- Now we need to evaluate the Christoffel symbol,

$$\Gamma^0{}_{\alpha\beta} = \frac{g^{00}}{2}\left[\frac{\partial g_{\alpha 0}}{\partial x^\beta} + \frac{\partial g_{\beta 0}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^0}\right] \tag{14.4.28}$$

with the metric $g_{\mu\nu}$ given by

$$g_{\mu\nu} = \begin{pmatrix} -1-2\Psi(x,t) & 0 & 0 & 0 \\ 0 & a^2[1+2\Phi(x,t)] & 0 & 0 \\ 0 & 0 & a^2[1+2\Phi(x,t)] & 0 \\ 0 & 0 & 0 & a^2[1+2\Phi(x,t)] \end{pmatrix} \tag{14.4.29}$$

Because of the symmetry in α and β, we can write the Christoffel symbol as

$$\Gamma^0{}_{\alpha\beta} = \frac{g^{00}}{2}\left[2\frac{\partial g_{\alpha 0}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^0}\right] \tag{14.4.30}$$

and from Eq. (14.4.29) we have $g_{00} = -1 - 2\Psi(x,t)$, so $g^{00} = -1 + 2\Psi(x,t)$. Furthermore, $x^0 \equiv t$, and therefore

$$\Gamma^0{}_{\alpha\beta} = \frac{-1+2\Psi}{2}\left[2\frac{\partial g_{\alpha 0}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial t}\right] \tag{14.4.31}$$

Going back to Eq. (14.4.27), we now include the factor $P^\alpha P^\beta/p$, and get

$$\Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{-1+2\Psi}{2}\left[2\frac{\partial g_{\alpha 0}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial t}\right]\frac{P^\alpha P^\beta}{p} \tag{14.4.32}$$

We first concentrate on the second term,

$$-\frac{\partial g_{\alpha\beta}}{\partial t}\frac{P^\alpha P^\beta}{p} = -\frac{\partial g_{00}}{\partial t}\frac{P^0 P^0}{p} - \frac{\partial g_{ij}}{\partial t}\frac{P^i P^j}{p} \tag{14.4.33}$$

$$= 2\frac{\partial \Psi}{\partial t}\frac{P^0 P^0}{p} - \frac{\partial g_{ij}}{\partial t}\frac{P^i P^j}{p} \tag{14.4.34}$$

Previously, (Eq. 4.14, p. 89) it was found that $P^0 = p/\sqrt{1+2\Psi}$, so the first term is

$$\frac{\partial \Psi}{\partial t}\frac{P^0 P^0}{p} = \frac{\partial \Psi}{\partial t}\frac{pp}{p(1+2\Psi)} \tag{14.4.35}$$

$$\approx \frac{\partial \Psi}{\partial t}p(1-2\Psi) \tag{14.4.36}$$

$$\approx \frac{\partial \Psi}{\partial t}p \tag{14.4.37}$$

where again the second-order term involving $(\partial\Psi/\partial t)\Psi$ has been dropped. For the second term we need the derivatives of the metric,

$$\frac{\partial g_{ij}}{\partial t} = \delta_{ij}\frac{\partial}{\partial t}a^2[1+2\Phi] \tag{14.4.38}$$

$$= \delta_{ij}\left(2a^2\frac{\partial \Phi}{\partial t} + (1+2\Phi)\frac{\partial a^2}{\partial t}\right) \tag{14.4.39}$$

$$= \delta_{ij}\left(2a^2\frac{\partial \Phi}{\partial t} + 2(1+2\Phi)a\dot{a}\right) \tag{14.4.40}$$

$$= \delta_{ij}\left(2a^2\frac{\partial \Phi}{\partial t} + 2a^2(1+2\Phi)\frac{\dot{a}}{a}\right) \tag{14.4.41}$$

$$= \delta_{ij}\left(2a^2\frac{\partial \Phi}{\partial t} + 2a^2(1+2\Phi)H\right) \tag{14.4.42}$$

$$= 2a^2\delta_{ij}\left(\frac{\partial \Phi}{\partial t} + H(1+2\Phi)\right) \tag{14.4.43}$$

(14.4.44)

We can now fill these results back into Eq. (14.4.34), and get

$$\boxed{-\frac{\partial g_{\alpha\beta}}{\partial t}\frac{P^\alpha P^\beta}{p} = 2\frac{\partial \Psi}{\partial t}p - 2a^2\delta_{ij}\left(\frac{\partial \Phi}{\partial t} + H(1+2\Phi)\right)\frac{P^i P^j}{p}} \tag{14.4.45}$$

which is Eq. (4.28) in the book.

To eliminate the factor $\delta_{ij}P^i P^j/p$, we use the result from (4.19),

$$P^i = p\hat{p}^i\frac{1-\Phi}{a} \tag{14.4.46}$$

i.e.

$$\delta_{ij}\frac{P^i P^j}{p} = p^2(1-2\Phi)/a^2/p = p(1-2\Phi)/a^2 \tag{14.4.47}$$

We now go back to Eq. (14.4.32) and can write

$$\Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{-1+2\Psi}{2}\left[2\frac{\partial g_{\alpha 0}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial t}\right]\frac{P^\alpha P^\beta}{p} \tag{14.4.48}$$

$$= \frac{-1+2\Psi}{2}\left[2\frac{\partial g_{\alpha 0}}{\partial x^\beta}\frac{P^\alpha P^\beta}{p} + 2\frac{\partial \Psi}{\partial t}p - 2a^2\left(\frac{\partial \Phi}{\partial t} + H(1+2\Phi)\right)p(1-2\Phi)/a^2\right] \tag{14.4.49}$$

$$= \frac{-1+2\Psi}{2}\left[2\frac{\partial g_{\alpha 0}}{\partial x^\beta}\frac{P^\alpha P^\beta}{p} + 2p\frac{\partial \Psi}{\partial t} - 2p\left(\frac{\partial \Phi}{\partial t} + H(1+2\Phi)\right)(1-2\Phi)\right] \tag{14.4.50}$$

Finally, we need to evaluate the sum

$$\frac{\partial g_{\alpha 0}}{\partial x^\beta}\frac{P^\alpha P^\beta}{p} \tag{14.4.51}$$

Because the metric is diagonal, this is non-zero only for $\alpha = 0$, so that

$$\frac{\partial g_{\alpha 0}}{\partial x^\beta}\frac{P^\alpha P^\beta}{p} = \frac{\partial g_{00}}{\partial x^\beta}\frac{P^0 P^\beta}{p} \tag{14.4.52}$$

$$= -2\frac{\partial \Psi}{\partial x^\beta}\frac{P^0 P^\beta}{p} \tag{14.4.53}$$

$$= -2\frac{\partial \Psi}{\partial x^\beta}(1-\Psi)P^\beta \tag{14.4.54}$$

$$= -2\frac{\partial \Psi}{\partial x^\beta}P^\beta \tag{14.4.55}$$

where, in the last step, we have again dropped the second-order term $(\partial\Psi/\partial x^\beta)\Psi$. Thus, we get Eq. (4.29):

$$\boxed{\Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{-1+2\Psi}{2}\left[-4\frac{\partial \Psi}{\partial x^\beta}P^\beta + 2p\frac{\partial \Psi}{\partial t} - 2p\left(\frac{\partial \Phi}{\partial t} + H(1+2\Phi)\right)(1-2\Phi)\right]} \tag{14.4.56}$$

We can simplify this further, by noting that $(1+2\Phi)(1-2\Phi) = 1-4\Phi^2 \approx 1$, and we can drop the second-order term $(\partial\Phi/\partial t)\Phi$, so that

$$\Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{-1+2\Psi}{2}\left[-4\frac{\partial \Psi}{\partial x^\beta}P^\beta + 2p\frac{\partial \Psi}{\partial t} - 2p\left(\frac{\partial \Phi}{\partial t} + H\right)\right] \tag{14.4.57}$$

We now need to finish evaluating the sum in the first term:

$$\frac{\partial \Psi}{\partial x^\beta}P^\beta = \frac{\partial \Psi}{\partial t}P^0 + \frac{\partial \Psi}{\partial x^i}P^i \tag{14.4.58}$$

$$= \frac{\partial \Psi}{\partial t}p(1-\Psi) + \frac{\partial \Psi}{\partial x^i}p\hat{p}^i\frac{1-\Phi}{a} \tag{14.4.59}$$

$$= \frac{\partial \Psi}{\partial t}p + \frac{\partial \Psi}{\partial x^i}\frac{p\hat{p}^i}{a} \tag{14.4.60}$$

$$\tag{14.4.61}$$

Inserting it in Eq. (14.4.57), we get

$$\Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{-1+2\Psi}{2}\left[-4\left(\frac{\partial \Psi}{\partial t}p + \frac{\partial \Psi}{\partial x^i}\frac{p\hat{p}^i}{a}\right) + 2p\frac{\partial \Psi}{\partial t} - 2p\left(\frac{\partial \Phi}{\partial t} + H\right)\right] \tag{14.4.62}$$

$$= (-1+2\Psi)\left[-p\frac{\partial \Psi}{\partial t} - 2\frac{\partial \Psi}{\partial x^i}\frac{p\hat{p}^i}{a} - p\left(\frac{\partial \Phi}{\partial t} + H\right)\right] \tag{14.4.63}$$

which is Eq. (4.30).

We can now insert this in Eq. (14.4.27),

$$
\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}t} &= p\left(\frac{\partial\Psi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right)-(-1+2\Psi)\left[-p\frac{\partial\Psi}{\partial t}-2\frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a}-p\left(\frac{\partial\Phi}{\partial t}+H\right)\right](1+2\Psi) && (14.4.64)\\
&= p\left(\frac{\partial\Psi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right)+(1-2\Psi)(1+2\Psi)\left[-p\frac{\partial\Psi}{\partial t}-2\frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a}-p\left(\frac{\partial\Phi}{\partial t}+H\right)\right] && (14.4.65)\\
&= p\left(\frac{\partial\Psi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right)-p\frac{\partial\Psi}{\partial t}-2\frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a}-p\left(\frac{\partial\Phi}{\partial t}+H\right) && (14.4.66)\\
&= p\frac{\partial\Psi}{\partial t}+\frac{p\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}-p\frac{\partial\Psi}{\partial t}-2\frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a}-p\left(\frac{\partial\Phi}{\partial t}+H\right) && (14.4.67)
\end{aligned}
$$

Simplifying further, we finally get

$$
\boxed{\frac{\mathrm{d}p}{\mathrm{d}t}=-p\left(H+\frac{\partial\Phi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right)} \qquad (14.4.68)
$$

which is the desired Eq. (4.32).

14.5 First order terms of the Boltzmann equation for photons

- We have

$$
\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\partial f}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i}-p\frac{\partial f}{\partial p}\left[H+\frac{\partial\Phi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] \qquad (14.5.1)
$$

and

$$
f\approx f^{(0)}-p\frac{\partial f^{(0)}}{\partial p}\Theta \qquad (14.5.2)
$$

We insert Eq. (14.5.2) in Eq. (14.5.1) and get

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial}{\partial t}\left(f^{(0)}-p\frac{\partial f^{(0)}}{\partial p}\Theta\right)&\\
+\frac{\hat{p}^i}{a}\frac{\partial}{\partial x^i}\left(f^{(0)}-p\frac{\partial f^{(0)}}{\partial p}\Theta\right)&\\
-p\frac{\partial}{\partial p}\left(f^{(0)}-p\frac{\partial f^{(0)}}{\partial p}\Theta\right)\left[H+\frac{\partial\Phi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right]& \qquad (14.5.3)
\end{aligned}
$$

On the first line, we can eliminate the zero-order term $\partial f^0/\partial t$, on the second line the derivative $\partial f^0/\partial x^i$ vanishes (since $f^{(0)}$ does not depend on x^i), and the zero-order term $Hp\partial f^0/\partial p$ (third line) can also be eliminated (since we are only interested in the first order terms). The remaining terms are

$$
\begin{aligned}
\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_1 = -p\frac{\partial}{\partial t}\left(\frac{\partial f^{(0)}}{\partial p}\Theta\right)-p\frac{\hat{p}^i}{a}\frac{\partial}{\partial x^i}\left(\frac{\partial f^{(0)}}{\partial p}\Theta\right)&\\
-p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Phi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right]+p\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\Theta\right)\left[H+\frac{\partial\Phi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right]& \qquad (14.5.4)
\end{aligned}
$$

In the second term on the last line, the terms proportional to $\Theta\frac{\partial\Phi}{\partial t}$ and $\Theta\frac{\partial\Psi}{\partial x^i}$ are second-order and can be eliminated. Since f^0 does not depend on x^i, the $\frac{\partial}{\partial x^i}\frac{\partial f^{(0)}}{\partial p}\Theta$ term simplifies to $\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial x^i}$ Then we are left with:

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_1 = p\frac{\partial}{\partial t}\left(\frac{\partial f^{(0)}}{\partial p}\Theta\right) - p\frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}\left(\frac{\partial f^{(0)}}{\partial p}\right) - p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] + Hp\Theta\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right) \quad (14.5.5)$$

This is Eq. (4.40) in the book.

- We are comparing

$$Hp\Theta\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right) \quad (14.5.6)$$

and

$$p\Theta\frac{\mathrm{d}T/\mathrm{d}t}{T}\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right) \quad (14.5.7)$$

From Eq. (4.38) in the book (that follows from looking at the zero-order terms) we have that

$$\frac{\mathrm{d}T/\mathrm{d}t}{T} = -\frac{\mathrm{d}a/\mathrm{d}t}{a} = -\frac{\dot{a}}{a} \equiv -H \quad (14.5.8)$$

which gives the desired result.

14.6 Exercise 5, Chapter 4

Why the factor $1/p$ in front of the equation for the collision terms (Eq. 4.45 in the book)? This is **Exercise 5, p. 114**: In GR, it would be more appropriate to write the derivative of f in terms of the affine parameter, λ:

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = C' \quad (14.6.1)$$

whereas we use the time derivative explicitly (Eq. 4.1):

$$\frac{\mathrm{d}f}{\mathrm{d}t} = C \quad (14.6.2)$$

Using the implicit definition of λ via the four-momentum, we can go from one to the other using

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = \frac{\mathrm{d}f}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{\mathrm{d}f}{\mathrm{d}t}P^0 = \frac{\mathrm{d}f}{\mathrm{d}t}p(1-\Psi) \quad (14.6.3)$$

That is,

$$C' = Cp(1-\Psi) \quad (14.6.4)$$

or, dropping the term $C\Psi$ since both factors are first-order terms (zero in equilibrium) and the product therefore second-order:

$$C = C'/p \quad (14.6.5)$$

This explains the factor $1/p$.

14.7 The Einstein tensor in the perturbed FRW metric

- We start with $\mu = \nu = 0$:

$$\Gamma^0{}_{00} = \frac{1}{2} g^{0\alpha} \left[g_{\alpha 0,0} + g_{\alpha 0,0} - g_{00,\alpha} \right] \tag{14.7.1}$$

Because the metric is diagonal, only terms the terms multiplied by g^{00} are non-zero. Since Ψ is a small perturbation, we have

$$g^{00} = 1/g_{00} \sim -1 + 2\Psi \tag{14.7.2}$$

so

$$\begin{aligned}
\Gamma^0{}_{00} &= \frac{-1+2\Psi}{2} \left[g_{00,0} + g_{00,0} - g_{00,0} \right] & (14.7.3)\\
&= \frac{-1+2\Psi}{2} (g_{00,0}) & (14.7.4)\\
&= \frac{-1+2\Psi}{2} \frac{\partial(-1-2\Psi)}{\partial t} & (14.7.5)\\
&= \frac{-1+2\Psi}{2} (-2\Psi_{,0}) & (14.7.6)\\
&\approx \Psi_{,0} & (14.7.7)
\end{aligned}$$

where, in the last step, we have as usual dropped the second-order term $\Psi\Psi_{,0}$.

- Then we go on to look at one of the two indices μ or ν being spatial, while the other remains 0 (time). We thus evaluate

$$\Gamma^0{}_{\mu 0} = \frac{1}{2} g^{0\alpha} \left[g_{\alpha\mu,0} + g_{\alpha 0,\mu} - g_{\mu 0,\alpha} \right] \tag{14.7.8}$$

(we could also have started with $\mu = 0$ and ν spatial; because of the symmetry the result is the same). Again, only the terms with $\alpha = 0$ count, so

$$\begin{aligned}
\Gamma^0{}_{i0} &= \frac{1}{2} g^{00} \left[g_{0i,0} + g_{00,i} - g_{i0,0} \right] & (14.7.9)\\
&= \frac{-1+2\Psi}{2} \left[g_{0i,0} + g_{00,i} - g_{i0,0} \right] & (14.7.10)
\end{aligned}$$

where i as usual refers to a spatial index. Because of the symmetry, the first and last terms cancel so

$$\begin{aligned}
\Gamma^0{}_{i0} &= \frac{-1+2\Psi}{2} \left[g_{00,i} \right] & (14.7.11)\\
&= \frac{-1+2\Psi}{2} (-2\Psi_{,i}) & (14.7.12)
\end{aligned}$$

Eliminating again the second-order term $\Psi\Psi_{,i}$, we get

$$\Gamma^0{}_{i0} = \Gamma^0{}_{0i} = \Psi_{,i} \tag{14.7.13}$$

Finally, moving to Fourier space, the spatial derivatives are

$$\tilde{\Psi}_{,i} = ik_i \tilde{\Psi} \tag{14.7.14}$$

where k_i is the wave number corresponding to the component i.

- Both indices spatial: As usual, terms with $\alpha \neq 0$ are zero, so

$$\Gamma^0{}_{ij} = \frac{1}{2} g^{00} \left[g_{0i,j} + g_{0j,i} - g_{ij,0} \right] \tag{14.7.15}$$

Here, the two first terms in the brackets are zero (because the metric is diagonal), and g^{00} is the same as in the other cases, so

$$\begin{aligned}
\Gamma^0{}_{ij} &= \frac{-1+2\Psi}{2} \left[-g_{ij,0} \right] && (14.7.16)\\
&= \frac{1-2\Psi}{2} \left[\delta_{ij} \frac{\partial a^2 [1+2\Phi]}{\partial t} \right] && (14.7.17)\\
&= \frac{1-2\Psi}{2} \delta_{ij} \left[2a\dot{a}[1+2\Phi] + 2a^2 \Phi_{,0} \right] && (14.7.18)\\
&= \delta_{ij}(1-2\Psi)a^2 \left[a\dot{a}/a^2 [1+2\Phi] + \Phi_{,0} \right] && (14.7.19)\\
&= \delta_{ij}(1-2\Psi)a^2 \left[H[1+2\Phi] + \Phi_{,0} \right] && (14.7.20)\\
&= \delta_{ij} a^2 \left[H[1+2\Phi] + \Phi_{,0} - 2\Psi H[1+2\Phi] - 2\Psi\Phi_{,0} \right] && (14.7.21)\\
&\approx \delta_{ij} a^2 \left[H[1+2\Phi] + \Phi_{,0} - 2\Psi H \right] && (14.7.22)
\end{aligned}$$

and finally

$$\boxed{\Gamma^0{}_{ij} = \delta_{ij} a^2 \left[H + 2H(\Phi - \Psi) + \Phi_{,0} \right]} \tag{14.7.23}$$

which is Eq. (5.6) in the book.

Solutions, Cosmology 2016/2016, Week 15

15.1 Momenta of the photon perturbations

Show that

$$\int_{-1}^{1} d\mu\mu^2\Theta(\mu) = \frac{2}{3}\Theta_0 - \frac{4}{3}\Theta_2 \tag{15.1.1}$$

The second moment (Θ_2) is defined as

$$\begin{aligned}
\Theta_2 &= -\frac{1}{2}\int_{-1}^{1} d\mu\frac{3\mu^2-1}{2}\Theta(\mu) && (15.1.2)\\
&= -\frac{1}{2}\left(\int_{-1}^{1} d\mu\frac{3}{2}\mu^2\Theta(\mu) - \int_{-1}^{1} d\mu\frac{1}{2}\Theta(\mu)\right) && (15.1.3)\\
&= -\frac{3}{4}\int_{-1}^{1} d\mu\mu^2\Theta(\mu) + \frac{1}{4}\int_{-1}^{1} d\mu\Theta(\mu) && (15.1.4)\\
\frac{4}{3}\Theta_2 &= -\int_{-1}^{1} d\mu\mu^2\Theta(\mu) + \frac{1}{3}\int_{-1}^{1} d\mu\Theta(\mu) && (15.1.5)
\end{aligned}$$

so we have

$$\begin{aligned}
\int_{-1}^{1} d\mu\mu^2\Theta(\mu) &= \frac{1}{3}\int_{-1}^{1} d\mu\Theta(\mu) - \frac{4}{3}\Theta_2 && (15.1.6)\\
&= \frac{2}{3}\Theta_0 - \frac{4}{3}\Theta_2 && (15.1.7)
\end{aligned}$$

15.2 From inhomogeneities to anisotropies (I)

- This follows from straight forward evaluation, remembering that Θ also depends on η:

$$\begin{aligned}
e^{-ik\mu\eta+\tau}\frac{d}{d\eta}\left[\Theta e^{ik\mu\eta-\tau}\right] &= e^{-ik\mu\eta+\tau}\left[\dot{\Theta}e^{ik\mu\eta-\tau} + \Theta(ik\mu-\dot{\tau})e^{ik\mu\eta-\tau}\right] && (15.2.1)\\
&= \dot{\Theta} + (ik\mu-\dot{\tau})\Theta && (15.2.2)
\end{aligned}$$

- We multiply both sides by $e^{ik\mu\eta-\tau}$:

$$\begin{aligned}
e^{-ik\mu\eta+\tau}e^{ik\mu\eta-\tau}\frac{d}{d\eta}\left[\Theta e^{ik\mu\eta-\tau}\right] &= \tilde{S}\,e^{ik\mu\eta-\tau} && (15.2.3)\\
\frac{d}{d\eta}\left[\Theta e^{ik\mu\eta-\tau}\right] &= \tilde{S}\,e^{ik\mu\eta-\tau} && (15.2.4)
\end{aligned}$$

  and then integrate over η from $\eta_{\rm init}$ to η_0 (today):

$$\begin{aligned}
\int_{\eta_{\rm init}}^{\eta_0} d\eta\frac{d}{d\eta}\left[\Theta e^{ik\mu\eta-\tau}\right] &= \int_{\eta_{\rm init}}^{\eta_0} d\eta\tilde{S}\,e^{ik\mu\eta-\tau} && (15.2.5)\\
\Theta(\eta_0)e^{ik\mu\eta_0-\tau} - \Theta(\eta_{\rm init})e^{ik\mu\eta_{\rm init}-\tau} &= \int_{\eta_{\rm init}}^{\eta_0} d\eta\tilde{S}\,e^{ik\mu\eta-\tau} && (15.2.6)\\
\Theta(\eta_0)e^{ik\mu\eta_0-\tau} &= \Theta(\eta_{\rm init})e^{ik\mu\eta_{\rm init}-\tau} + \int_{\eta_{\rm init}}^{\eta_0} d\eta\tilde{S}\,e^{ik\mu\eta-\tau} && (15.2.7)
\end{aligned}$$

$$\Theta(\eta_0) = \Theta(\eta_{\text{init}})e^{ik\mu\eta_{\text{init}}-\tau}e^{-ik\mu\eta_0+\tau} + e^{-ik\mu\eta_0+\tau}\int_{\eta_{\text{init}}}^{\eta_0} \mathrm{d}\eta\tilde{S}\, e^{ik\mu\eta-\tau} \tag{15.2.8}$$

At η_0 = today, we have $\tau(\eta_0) = 0$ (per definition). Then

$$\begin{aligned}\Theta(\eta_0) &= \Theta(\eta_{\text{init}})e^{ik\mu\eta_{\text{init}}-\tau}e^{-ik\mu\eta_0} + e^{-ik\mu\eta_0}\int_{\eta_{\text{init}}}^{\eta_0} \mathrm{d}\eta\tilde{S}\, e^{ik\mu\eta-\tau} &(15.2.9)\\ &= \Theta(\eta_{\text{init}})e^{ik\mu(\eta_{\text{init}}-\eta_0)}e^{-\tau(\eta_{\text{init}})} + e^{-ik\mu\eta_0}\int_{\eta_{\text{init}}}^{\eta_0} \mathrm{d}\eta\tilde{S}\, e^{ik\mu\eta-\tau} &(15.2.10)\end{aligned}$$

$$\boxed{\Theta(\eta_0) = \Theta(\eta_{\text{init}})e^{ik\mu(\eta_{\text{init}}-\eta_0)}e^{-\tau(\eta_{\text{init}})} + \int_{\eta_{\text{init}}}^{\eta_0} \mathrm{d}\eta\tilde{S}\, e^{ik\mu(\eta-\eta_0)-\tau(\eta)}} \tag{15.2.11}$$

which is Eq. (8.45). If η_{init} is very early, then the optical depth $\tau(\eta_{\text{init}}) \gg 1$ and the first term vanishes:

$$\Theta(\eta_0) \simeq \int_{\eta_{\text{init}}}^{\eta_0} \mathrm{d}\eta\tilde{S}\, e^{ik\mu(\eta-\eta_0)-\tau(\eta)} \tag{15.2.12}$$

In other words, the initial perturbations do not affect the visible anisotropies. For the same reason, it makes no difference if we integrate from $\eta = 0$ or some time very soon thereafter, so we can set the lower limit of the integral to 0:

$$\boxed{\Theta(\eta_0) \simeq \int_{0}^{\eta_0} \mathrm{d}\eta\tilde{S}\, e^{ik\mu(\eta-\eta_0)-\tau(\eta)}} \tag{15.2.13}$$

15.3 From inhomogeneities to anisotropies (II)

Hand-in