Binary evolution in a nutshell
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1 Stellar timescales

1.1 Dynamical timescale

An estimate of the dynamical timescale of a star can be given by estimating how long it would take a star of given mass and radius to collapse. This is also known as the free-fall timescale:

$$\tau_{\text{dyn}} \sim \tau_{\text{ff}} \sim \sqrt{\frac{R^3}{GM}} \approx 1.6 \times 10^3 \text{s} \sqrt{(\frac{R}{R_\odot})^3} \approx 5.1 \times 10^{-5} \text{yr} \sqrt{(\frac{R}{R_\odot})^3} \left(\frac{M}{M_\odot}\right). \quad (1)$$

1.2 Thermal timescale

The thermal timescale is the timescale at which a star with given mass, radius and luminosity could radiate away its gravitational energy. This is often approximated by the Kelvin-Helmholtz timescale:

$$\tau_{\text{th}} \sim \tau_{\text{KH}} \sim \frac{GM^2}{RL} \approx 3.1 \times 10^7 \text{ yr} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right) \left(\frac{L}{L_\odot}\right). \quad (2)$$

For giants (stars with a clear core-envelope structure), one often uses:

$$\tau_{\text{th}} \sim \tau_{\text{KH}} \sim \frac{GM M_{\text{env}}}{RL} \approx 3.1 \times 10^7 \text{ yr} \left(\frac{M}{M_\odot}\right) \left(\frac{M_{\text{env}}}{M_\odot}\right) \left(\frac{R}{R_\odot}\right) \left(\frac{L}{L_\odot}\right). \quad (3)$$

1.3 Nuclear-evolution timescale

For main-sequence stars with masses between roughly $0.2 M_\odot$ and $30 M_\odot$, the empirical mass-luminosity relation

$$\left(\frac{L}{L_\odot}\right) \approx \left(\frac{M}{M_\odot}\right)^{3.8} \quad (4)$$

seems to hold reasonably well. Scaling this relation with the nuclear-evolution timescale of the Sun gives an approximation of the nuclear-evolution timescale of a star with given mass:

$$\tau_{\text{nuc}} \sim \left(\frac{M}{M_\odot}\right) \left(\frac{L}{L_\odot}\right)^{-1} \tau_{\text{nuc,\odot}} \approx 10^{10} \text{ yr} \left(\frac{M}{M_\odot}\right)^{-2.8}. \quad (5)$$

Here we made the assumption that the fraction of the stellar mass available for hydrogen fusion is the same for all stars. In reality, this factor will vary; it is $\sim 0.15$ for the Sun, but may be much larger for more massive stars, for example because they have convective cores and can mix material from outside into the ‘burning’ region to be fused.

For evolutionary phases other than the main sequence, one could scale Eq.5 with the energy efficiency of the nuclear reaction involved. However, this is only true for other core-burning phases, in which case a burning shell often provides a significant fraction (if not the majority) of the luminosity.

If the timescale is needed in the context of mass transfer, the relevant timescale is often that of the rate $\dot{R}$ of change of the stellar radius due to the nuclear evolution of the star. If a stellar-evolution code is available, so that $R$ and hence $\dot{R}$ can be computed, one can use

$$\tau_{\text{nuc}} \approx \frac{R}{\dot{R}}. \quad (6)$$

Of course, this number is only meaningful if stellar evolution is indeed the drive for the change in radius. It is useless if for instance the star fills its Roche lobe.

2 Geometry of a binary

See Figure 1 for a sketch of the geometry. For most expressions a circular orbit is assumed. The subscript $i$ denotes a binary component ($i = 1, 2$); the subscript $(3 - i)$ indicates the other binary component.
Figure 1: Schematic geometry of a binary with masses $M_1 > M_2$, and orbital separation $a$. The distances $a_1, a_2$ are from the binary members to the centre of mass ($cm$).

2.1 Masses

The total mass $M_T$, mass ratio $q$ and reduced mass $\mu$ of a binary are given by:

$$M_T = M_1 + M_2,$$

$$q_i = \frac{M_i}{M_{(3-i)}},$$

and

$$\mu = \frac{M_1M_2}{M_T}.$$ (9)

When working with gravitational waves, the chirp mass $\mathcal{M}$ and symmetric mass ratio $0 < \eta \leq 1$ are often used:

$$\eta = \frac{M_1M_2}{M_T^2} = \frac{q}{(1+q)^2};$$

$$\mathcal{M} = M_T \eta^{3/5}. $$ (11)

To compute $0 < q \leq 1$ from $\eta$, use

$$q = \frac{1 - \sqrt{1 - 4\eta}}{1 + \sqrt{1-4\eta}}.$$ (12)

2.2 Centre of mass

The distance to the centre of mass ($cm$) for each of the binary members ($a_i$, see Fig. 1) is given by:

$$a_i = \frac{M_{(3-i)}}{M_T} a$$

$$a_1 = \frac{a_2}{M_2} = \frac{a}{M_1} = \frac{a}{M_T}$$

$$M_1a_1 = M_2a_2 = \mu a$$ (15)

2.3 Kepler’s law

Kepler’s law can be written as:

$$\left(\frac{2\pi}{P}\right)^2 = \omega^2 = \frac{GM_T}{a^3}$$ (16)

$$P = \left(\frac{4\pi^2}{GM_T}\right)^{1/2} a^{3/2}$$ (17)

$$a = \left(\frac{GM_T}{4\pi^2}\right)^{1/3} P^{2/3}$$ (18)
The Roche potential

The gravitational and rotational potential in a frame corotating with the binary is called the Roche potential (see Fig. 2), and defined as:

$$\Phi_{\text{Roche}}(\vec{r}) = -\frac{GM_1}{|\vec{r} - \vec{r}_1|} - \frac{GM_2}{|\vec{r} - \vec{r}_2|} - \frac{1}{2} (\vec{\omega} \times \vec{r})^2,$$

$$\vec{F} = -\nabla \Phi_{\text{Roche}}. \tag{19}$$

Note that, because we’re in a corotating frame:

$$\lim_{r \to \infty} -\frac{1}{2} (\vec{\omega} \times \vec{r})^2 = -\infty.$$ 

This is fine, because this applies to the corotating frame, which does not extend far beyond the binary.

Note also that in a co-rotating frame, moving particles will experience the fictitious Coriolis force:

$$\vec{F}_c = -2m (\vec{\omega} \times \vec{v}). \tag{21}$$

3.1 Roche-lobe radius

An approximation for the Roche-lobe radius accurate within 2% for $0 < q_i < 0.8$ is given by Paczyński (1971) (note that there is a typo in this equation in Paczyński, 1967):

$$R_{\text{RL,1}} \approx \frac{2}{3^{4/3}} \left( \frac{M_i}{M_T} \right)^{1/3}. \tag{22}$$

An approximation accurate (the first part) within 1% for $0 < q_i < \infty$ was derived by Eggleton (1983) (see Fig. 3):

$$R_{\text{RL,1}} \approx \frac{0.49 q_i^{2/3}}{0.6 q_i^{2/3} + \ln \left(1 + q_i^{1/3}\right)} \approx \frac{0.44 q_i^{0.33}}{(1 + q_i)^{0.24}}. \tag{23}$$
For a binary in which star 1 fills its Roche lobe:

\[ P_{\text{orb}} \approx 0.35 \left( \frac{R_1}{M_1} \right)^{1/2} \left( \frac{2}{1 + q_1} \right)^{0.2} \text{ days.} \]  

In general,

\[ P_{\text{orb}} \propto (\rho_1)^{-1/2}. \]  

### 4 Orbital energy and angular momentum

#### 4.1 Orbital energy

The orbital energy of a circular binary is given by:

\[ E_{\text{orb}} = E_{\text{pot}} + E_{\text{kin}} = -\frac{G M_1 M_2}{a} + \frac{G M_1 M_2}{2a} = -\frac{G M_1 M_2}{2a}. \]  

The “stellar” unit for energy is

\[ \frac{G M_1^2}{R_\odot} \approx 3.79 \times 10^{48} \text{ erg.} \]  

#### 4.2 Orbital angular momentum

The angular momentum (AM) of binary component \( i \) in a circular orbit:

\[ J_i = |\vec{J}_i| = M_i |\vec{v}_i \times \vec{a}_i| = M_i v_i a_i \]

\[ = M_i a_i^2 \omega \]

\[ J = J_1 + J_2 = \mu a^2 \omega \]  

The total orbital angular momentum \( J \) can be written in terms of the orbital period \( P \), orbital separation \( a \), or both:

\[ J(a, P) = \mu a^2 \frac{2\pi}{P}; \]

\[ J(P) = G^{2/3} \frac{M_1 M_2}{M_T^{1/3}} \left( \frac{P}{2\pi} \right)^{1/3}; \]

\[ J(a) = M_1 M_2 \left( \frac{G a}{M_T} \right)^{1/2} = \mu \sqrt{G M_T a}. \]
The “stellar” unit for angular momentum is
\[ G^{1/2} M^{3/2} R^{1/2} \approx 6.05 \times 10^{51} \text{ erg s.} \] (34)

Inversely, the orbital period and separation can be computed from the angular momentum:
\[ P = \left( \frac{J}{\mu} \right)^3 \frac{2\pi}{G^2 M_T^2} = 2\pi \frac{M_T}{G^2} \left( \frac{J}{M_1 M_2} \right)^3 ; \] (35)
\[ a = \frac{J^2}{\mu^2 G M_T} = \left( \frac{J}{M_1 M_2} \right)^2 \frac{M_T}{G} . \] (36)

The orbital angular momentum for binary component \( i \) is given by:
\[ J_i = J M_i (3 - i) M_T = \mu J M^2_i . \] (37)

The specific orbital angular momentum for binary component \( i \) can be written as:
\[ h_i = \frac{J_i}{M_i} = \mu \frac{J}{M^2_i} . \] (38)

5 Orbital angular-momentum loss

A general equation for angular-momentum loss from a binary can be obtained by taking the logarithmic derivatives of Eqs. 32 and 33:
\[ \frac{\dot{J}}{J} = \frac{\dot{M}_1}{M_1} + \frac{\dot{M}_2}{M_2} - \frac{1}{3} \frac{\dot{M}_T}{M_T} + \frac{1}{3} \frac{\dot{P}}{P} ; \] (39)
\[ = \frac{M_1}{M_1} + \frac{M_2}{M_2} \left( \frac{1}{2} \frac{\dot{M}_T}{M_T} + \frac{1}{2} \frac{\dot{a}}{a} . \right. \] (40)

5.1 Gravitational waves

Angular momentum lost due the emission of gravitational waves is given by:
\[ \left( \frac{dJ}{dt} \right)_{GW} = - \frac{32}{5} \frac{G^{7/2}}{c^5} \frac{M_T^2 M_2^2 M_1^{1/2}}{a^{7/2}} \] (41)
\[ = - \frac{32}{5} \frac{G^{7/3}}{c^5} \left( \frac{M_1 M_2}{M_T^{1/3}} \right)^2 \left( \frac{2\pi}{P} \right)^{7/3} \] (42)

(Peters, 1964). This can be expressed as a logarithmic derivative as:
\[ \left( \frac{\dot{J}}{J} \right)_{GW} = - \frac{32}{5} \frac{G^{5/3}}{c^5} \left( \frac{2\pi}{P} \right)^{8/3} \frac{M_1 M_2}{M_T^{1/3}} \] (43)
\[ = - \frac{32}{5} \frac{G^3 M_1 M_2 M_T}{a^4} . \] (44)

The timescale for angular-momentum loss due to GWs is then:
\[ \tau_{GW} = \left| \frac{J}{J} \right|_{GW} \approx 380 \text{ Gyr \( \left( 1 + q \right)^2 \left( \frac{M_T}{M_\odot} \right)^{-5/3} \left( \frac{P}{\text{day}} \right)^{8/3} , \right. \] (45)
and the time until contact or merger for a given detached binary is:
\[ t_{merge} = \frac{\tau_{GW}}{8} . \] (46)
5.2 Non-conservative mass transfer

If, during mass transfer in a binary, a fraction $\beta$ of the transferred mass is accreted ($\dot{M}_{(3-i)} = -\beta \dot{M}_i$) and a fraction $(1 - \beta)$ is lost from the binary with a fraction $\alpha$ of the specific angular momentum of the accretor, the angular momentum lost from the system is

$$
\left( \frac{dJ}{dt} \right)_{\text{MT}} = -\alpha (1 - \beta) a_{(3-i)}^2 \omega \dot{M}_i
$$

(47)

or

$$
\left( \frac{dJ}{dt} \right)_{\text{MT}} = -\alpha (1 - \beta) G^{2/3} \left( \frac{P}{2\pi} \right)^{1/3} \frac{M_i^2 \dot{M}_i}{M_T^{4/3}}.
$$

(48)

Written as a logarithmic derivative, this becomes

$$
\left( \frac{\dot{j}}{j} \right)_{\text{MT}} = -\alpha (1 - \beta) \frac{M_i}{M_{(3-i)}} \frac{\dot{M}_i}{M_T}.
$$

(49)

5.3 Magnetic braking

Magnetic braking removes angular momentum from a rotating star with a magnetic field. If such a star is in a binary, and (close to) Roche-lobe filling, tidal effects may force the star to corotate with the orbit, essentially removing this angular momentum from the binary orbit. A prescription for the angular-momentum loss due to magnetic braking is given by e.g. Verbunt & Zwaan (1981):

$$
\left( \frac{dJ}{dt} \right)_{\text{MB}} = -3.8 \times 10^{-30} M R^4 \omega^3 \text{ dyn cm.}
$$

(50)

For rapidly-rotating stars, it is assumed that the magnetic field strength may no longer increase with the rotational period of the star, “saturating” the magnetic braking. A prescription for saturated magnetic braking is given by e.g. Sills et al. (2000) (see van der Sluys et al., 2005, for a complete set of equations):

$$
\left( \frac{dJ}{dt} \right)_{\text{MB}} = -K \left( \frac{R}{R_\odot} \right)^{1/2} \left( \frac{M}{M_\odot} \right)^{-1/2} \omega^3, \quad \omega \leq \omega_{\text{crit}}
$$

(51)

$$
\left( \frac{dJ}{dt} \right)_{\text{MB}} = -K \left( \frac{R}{R_\odot} \right)^{1/2} \left( \frac{M}{M_\odot} \right)^{-1/2} \omega \omega_{\text{crit}}^2, \quad \omega > \omega_{\text{crit}}
$$

(52)

6 Spin

6.1 Black holes

In black holes and neutron stars, the spin is often indicated by a dimensionless quantity between zero and unity

$$
a_{\text{spin}} \equiv \frac{cJ}{GM^2},
$$

(53)

where $J$ is the spin angular momentum, $M$ the mass and $c$ and $G$ are the speed of light and Newton’s constant, respectively.

6.2 Neutron stars

For a theoretical spherical neutron star with constant density ($I = \frac{2}{5} M R^2$) we find

$$
a_{\text{spin}} = \frac{4\pi c}{5G} \frac{R^2}{M P} \approx 0.58 \left( \frac{R}{12 \text{ km}} \right)^2 \left( \frac{M}{1.4 M_\odot} \right)^{-1} \left( \frac{P}{\text{ms}} \right)^{-1}.
$$

(54)

Realistic neutron stars rotate at breakup speed around $a_{\text{spin}} \sim 0.7$ (Miller et al., 2011).
7 Mass transfer, mass loss

7.1 Drives of mass transfer

Mass transfer between the donor star and the accretor can be driven by a number of processes. The mass-transfer rate can be estimated roughly from the timescale of the appropriate process $\tau$:

$$\dot{M} \sim \frac{M_{\text{donor}}}{\tau}. \quad (55)$$

Mass transfer can be driven by intrinsic changes, i.e. changes in the donor star (see Sect. 7.2):

- dynamical instability of the donor. $\dot{M}_{\text{dyn}} \sim \frac{M_{\text{donor}}}{\tau_{\text{dyn}}}$ (see Sect. 1.1);
- thermal evolution of the donor: $\dot{M}_{\text{th}} \sim \frac{M_{\text{donor}}}{\tau_{\text{th}}}$ (see Sect. 1.2);
- nuclear evolution of the donor: $\dot{M}_{\text{nuc}} \sim \frac{M_{\text{donor}}}{\tau_{\text{nuc}}}$ (see Sect. 1.3);

Mass transfer may also be driven by extrinsic changes, i.e. loss of angular momentum, e.g.:

- gravitational waves;
- magnetic braking;
- mass loss from the system;
- tidal dissipation.

7.2 Stability of mass transfer

Once mass transfer (MT) commences, its stability depends on the response of the radius of the donor star and the response of the Roche-lobe radius to the mass loss of the donor. These responses are usually expressed using the logarithmic derivative and the Greek letter $\zeta$:

$$\zeta \equiv \left( \frac{d \log R}{d \log M} \right) \quad (56)$$

In order for the mass transfer to be stable, the donor star must remain within its Roche lobe; hence the radius of the donor star must shrink more rapidly or expand less rapidly than the radius of the donor’s Roche lobe during the MT:

$$\zeta_d \geq \zeta_{R_l}^1. \quad (57)$$

In the range of stability of mass transfer, there are three regimes:

1. MT is thermally stable, and proceeds on the nuclear-evolution timescale;
2. MT is thermally unstable, but dynamically stable, and proceeds on the thermal timescale;
3. MT is dynamically unstable, and proceeds on the dynamical timescale (as a common envelope).

In order to determine on which timescale the mass transfer occurs, one will have to test whether the MT will be thermally stable by comparing the Kelvin-Helmholtz mass-radius exponent $\zeta_{d,KH}$ to the Roche-lobe exponent $\zeta_{R_l}$. If this test indicates thermally unstable MT, one will have to do a second test, comparing the adiabatic exponent $\zeta_{d,ad}$ to $\zeta_{R_l}$.

The adiabatic response of a giant star’s radius to mass loss, based on a polytrope with $n = 3/2$, is given to 1% accuracy by

$$\zeta_{ad} = \frac{2}{3} \frac{m_c}{1 - m_c} - \frac{1}{3} \frac{1 - m_c}{1 + 2m_c} - 0.03m_c + 0.2 \frac{m_c}{1 + (1 - m_c)^{1/6}}, \quad (58)$$

where $m_c = M_{\text{He,ad}}/M_d$ is the mass fraction of the donor’s helium core (Soberman et al., 1997).

\footnote{In fact, violation of this inequality indicates that the mass-transfer rate increases. If this is only temporary, the mass transfer may still stabilise after an initial increase in $\dot{M}$.}
The response of the donor’s Roche-lobe radius to the mass transfer depends on the mass ratio of the binary \( q \equiv M_d/M_a \) and the accretion efficiency \( 0 \leq \beta \leq 1 \), defined as \( \beta = 1 \) for conservative (all mass lost from the donor is accreted by the secondary) and \( \beta = 0 \) as completely non-conservative MT:

\[
\zeta_R(q, \beta) = \frac{\partial \ln a}{\partial \ln M_d} \frac{\partial \ln (R_{\text{R}}/a)}{\partial \ln q} \frac{\partial \ln q}{\partial \ln M_d} = \frac{2M_d^2 - 2M_a^2 - M_dM_a(1-\beta)}{M_a(M_d+M_a)} + \left[ \frac{2}{3} - \frac{q^{1/3}}{3} \frac{1.2q^{1/3} + 1/(1+q^{1/3})}{0.6q^{2/3} + 1/(1+q^{1/3})} \right] \left[ 1 + \beta \frac{M_d}{M_a} \right].
\]

The second term in Eq. 60 comes from the approximation for the Roche-lobe radius by Eggleton (1983, see our Eq. 23). For a derivation, see Sect. 2.3 in Woods et al. (2012).

### 7.3 Stable, conservative mass transfer

If mass transfer (MT) is conservative, and ignoring stellar winds and other sources of angular-momentum loss, \( \dot{M}_2 = -\dot{M}_1, \dot{J}_T = 0 \) and \( \dot{J}_\text{wind} = 0 \).

Then the orbital evolution is described by:

\[
\frac{\dot{a}}{a} = 2\dot{M}_1 \left( \frac{M_1 - M_2}{M_1M_2} \right) = 2\dot{M}_1 (q-1)
\]

Note that
- if \( M_1 > M_2 \) and MT happens from star 1 to star 2, \( \dot{\theta} < 0 \) and the orbit shrinks;
- \( \dot{\theta} = 0 \) when \( M_1 = M_2 \), hence the orbit starts expanding when the mass ratio flips.

Then the orbital evolution is described by:

\[
\frac{a}{a_{\text{min}}} = \left( \frac{M_T^2}{4M_1M_2} \right)^2, \quad \text{where} \quad a_{\text{min}} = \frac{16J^2}{GM_T^3}.
\]

### 7.4 Stellar wind

Assume a fast (\( v_\text{wind} \gg v_\text{orb} \)), isotropic stellar wind from star 1: \( \dot{M}_1 = \dot{M}_T \) and \( \dot{M}_2 = 0 \). Then:

\[
\left( \frac{\dot{J}}{\dot{J}_\text{wind}} \right) = \frac{M_2 \dot{M}_T}{M_1 M_1} = \frac{1}{q_1} M_T M_T,
\]

and:

\[
\frac{\dot{a}}{a} = 2 \left( \frac{\dot{J}}{\dot{J}_\text{wind}} \right) = \frac{M_1 + 2M_2 \dot{M}_T}{M_1 M_1} = \frac{\dot{M}_T}{M_T}.
\]

Note that the result is independent of which star loses the mass and is generally applicable to wind mass loss. Hence:

\[
\frac{\dot{a}}{a} = -\frac{\dot{M}}{M}; \quad a \propto M_T^{-1};
\]

\[
\frac{\dot{P}}{P} = -2\frac{\dot{M}}{M}; \quad P_{\text{orb}} \propto M_T^{-2}.
\]

### 7.5 Eddington limit

When a compact object (e.g. NS, BH) accretes matter, it radiates away energy produced from the infalling matter. If this luminosity becomes sufficiently high, the radiation pressure may prevent further accretion. Hence, there is a maximum accretion rate for an accretor, known as the Eddington limit.
We assume that the accreted matter is a plasma and each “particle” has the mass of a proton $m_p$ and the Thomson cross section of an electron $\sigma_T$. If the luminosity force cancels out gravity on such a particle:

$$ F_L = F_g \rightarrow \frac{L}{\epsilon} \frac{\sigma_T}{4\pi r^2} = \frac{GM_\star m_p}{r^2}, $$

so that the *Eddington luminosity* ($L_{\text{edd}}$) is defined as:

$$ L_{\text{edd}} = \frac{4\pi c G m_p}{\sigma_T} \approx 3.3 \times 10^4 \left( \frac{M_\star}{M_\odot} \right) L_\odot, $$

and the *Eddington accretion limit* ($\dot{M}_{\text{edd}}$) as:

$$ \dot{M}_{\text{edd}} = \frac{4\pi c m_p}{\sigma_T} R_\star \approx 1.5 \times 10^{-8} \left( \frac{R_\star}{10 \text{ km}} \right) M_\odot \text{ yr}^{-1}. $$

### 7.6 Classical common envelope

A common envelope (CE) is used to explain many observed compact binaries (Paczynski, 1976). A CE is initiated when mass transfer is dynamically unstable and hence (the initial phase of) a CE takes place on the short dynamical timescale. The orbital change due to a classical common envelope is described by energy balance between the orbit and the binding energy of the donor’s envelope:

$$ E_{\text{bind,env}} = \alpha_{\text{CE}} \Delta E_{\text{orb}} = \alpha_{\text{CE}} \left[ \frac{GM_1 M_2}{2a_i} - \frac{GM_1 M_2}{2a_f} \right] $$

(Webbink, 1984).

In many papers and especially population-synthesis codes, the envelope binding energy is approximated by

$$ E_{\text{bind,env}} \approx \frac{GM_{\text{env}} M_\star}{\lambda_{\text{env}} R_\star}, $$

where $\lambda_{\text{env}}$ is the *envelope structure parameter*, often assumed to be constant. However, see Dewi & Tauris (2000); Tauris & Dewi (2001); van der Sluys et al. (2006, 2010) on values for $\lambda_{\text{env}}$, and Loveridge et al. (2010) on how to approximate $E_{\text{bind,env}}$ (so that you no longer need $\lambda_{\text{env}}$).

Still, much of the uncertainty in $E_{\text{bind,env}}$ comes from the discussion of which energy sources should be included (in particular, can the recombination energy actually be used to expel the envelope), and, for massive stars ($M \gtrsim 5 \sim 8 M_\odot$), from the uncertainty of which fraction of the star is left after the CE (*i.e.*, the definition of the core mass).

### 7.7 Envelope ejection based on AM conservation

Nelemans et al. (2000) showed that a combination of stable mass transfer and a classical CE cannot explain observed double white dwarfs (DWDs). They suggest a mode of envelope ejection (EE) where angular-momentum, rather than energy, balance is important:

$$ \frac{J_f - J_i}{J_i} = \gamma \frac{M_1 f - M_1 i}{M_{\text{tot}, i}}, $$

where $\gamma$ describes how many times the specific angular momentum of the *system* is carried away by the mass loss. Nelemans et al. (2000, 2001); Nelemans & Tout (2005) show that this works well for observed (populations of) DWDs for $\gamma \approx 1.5 \sim 1.75$. However, a clear physical picture of why this works and why $\gamma$ should have these or any other values is currently lacking.

van der Sluys et al. (2006) used this idea and designed similar prescriptions for an envelope ejection with the specific angular momentum of the *donor* ($\gamma_d$, e.g. a strong stellar wind) and *accretor* ($\gamma_a$, isotropic wind from the accretor) respectively:

$$ \frac{J_f - J_i}{J_i} = \gamma_d \frac{M_1 f - M_1 i M_2 i}{M_{\text{tot}, f}}; $$

$$ \frac{J_f - J_i}{J_i} = \gamma_a \left[ \frac{M_{\text{tot}, i}}{M_{\text{tot}, f}} \exp \left( \frac{M_1 f - M_1 i}{M_2} \right) - 1 \right]. $$
They show that a $\gamma_d$-EE for the formation of the first WD followed by a $\gamma_a$-EE for the formation of the second WD explains observed values of $P$, $q$ and the difference in cooling age for DWDs well, for $\gamma_d, \gamma_a \approx 1.0$.

The $\gamma$-EE scenarios do not have to occur on the dynamical timescale, but must happen on a timescale that is short compared to the nuclear-evolution timescale (since the companion does not accrete, and the donor’s core mass does not grow).

7.8 Darwin instability

Apart from Roche-lobe overflow, an CE may be initiated by the Darwin instability (Darwin, 1879). If the spin angular momentum of one of the binary components exceeds $\frac{3}{2} J_{\text{orb}}$, the star is tidally locked to the orbit and the star expands (due to evolution), a stable binary orbit is no longer possible, and the two stars are bound to plunge into each other. When the star expands, it needs to spin up by extracting AM from the orbit, which then spins up itself, so that the star needs more AM to keep pace, et cetera. This will result in an “AM catastrophe”, an orbital spiral-in and a CE.

References

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