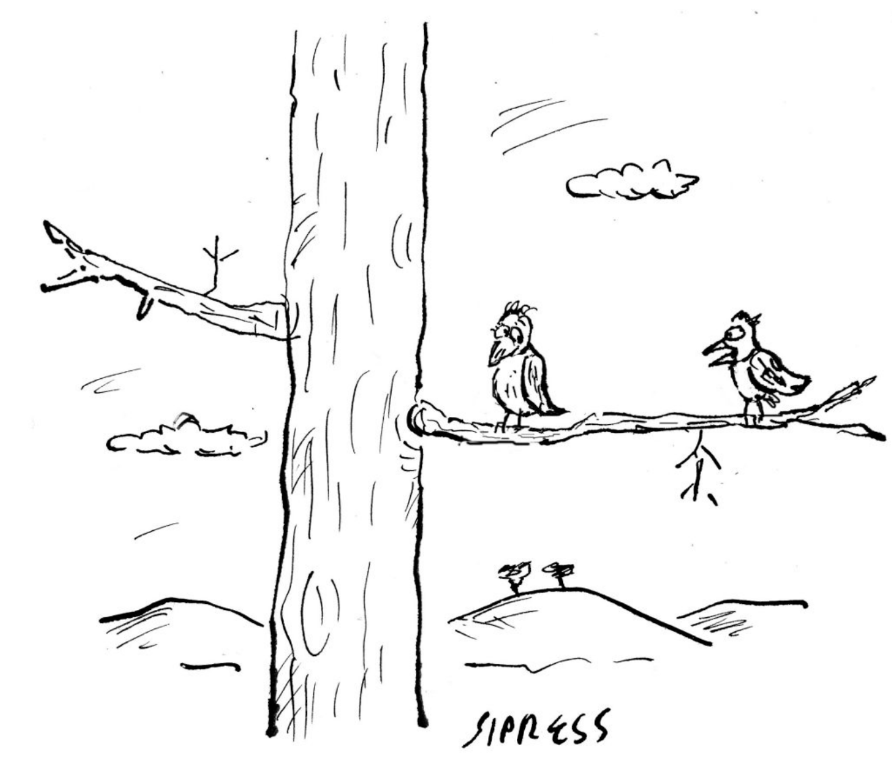
Intro to GR and GW astronomy

Samaya Nissanke
Radboud University, Nijmegen



"Was that you I heard just now, or was it two black holes colliding?"

Part I: BHs!!!

Plan: Part I - BHs, intro, history, zoo of BHs (~ how they form)

Part II - Schwarzschild metric:

in particular, orbits for particles (& light but not covered here)

$\Rightarrow$ critical for how we detect BHs.

Part III - Kerr BHs - spin BHs (most astrophysical BHs should have spin)

$\Rightarrow$ provides immense energy source

Literature + References: Chps. 5 + 12, Shapiro & Teukolsky.

Goodman notes on high energy astro } web.
Cole Miller " " " " "

[$G = c = 1$; Mass - M ; Length - $M = \frac{GM}{c^2}$ (1.5 km) ; time $M = \frac{GM}{c^3}$ $(5\times10^{-6}\mathrm{s})$]

PART I

Introduction.

1) Defn: $\boxed{\text{Black Holes :}}$ objects with an event horizon compared to a material surface.

Events within horizon cannot be seen by external observer.

→ end points of stellar evolution → enough mass from surrounding ☆ falls back onto Fe core so that $M_{chand.}$ (NS) is exceeded → BH!

→ power most luminous objs. (AGN + quasars).

→ simplest macroscopic objects → for a general stationary voln.

→ only ∃ characterisable observables

* mass M

* angular momentum J

(* charge Q).

⇒ Wheeler (1968) coined BHs have 'no hair': grav. waves will

2) Concept of Event Horizon.

↓

no light can escape

⇒ mainly observable by dynamics of ☆s around supermassive BH or accretion of infalling matter +/jets →

huge energy output!

Heuristically, escape velocity for object M and radius R

is $\frac{1}{2} v^2 = \frac{GM}{R} \geq c^2$ if $R < \frac{2GM}{c^2}$

[Laplace 1795, Mitchell 1793].

(cf. $v_{esc} \sim 10$ km/s $\oplus$

$v_{esc} \sim 700$ km/s $\odot$).

In GR, $R_s \sim \frac{2GM}{c^2}$ or $R_s \simeq 3\text{km} \left(\frac{M}{M_\odot} \right)$

[$\oplus$: 1cm].

(3) Brief History of BHs :

Theory -- i) Laplace (1795) noticed in Newtonian gravity + Newton's corpusular theory of light → light cannot escape from an object of sufficiently large M and small R

[also : John Michell (1783) hypothesis → ☆ $\bar{c}$ $250 \times R_\odot$ and average density equal to ⊕ would be dark !].

ii) Schwarzchild (1916) : Sol^{n} to Einstein's field eqns for the gravitational fields surrounding spherical mass.

iii) Chandrasekhar (1931) : Upper mass limit to completely degenerate config.

⇒ Eddington (1935) realised implications for massive ☆ collapse.

iv) Landau ; Oppenheimer + Snyder

v) Wheeler (1968) : BH ~ problem of gravit. collapse

⇒ name of BHs !

— * — * —

· Observations – in parallel, discovery of quasars (1963).

- pulsars (1968)
- compact X-ray sources (1962)

esp. Cygnus X-1 (1970) - blue SG (~30 $M_\odot$) orbiting a ~10 $M_\odot$ BH

- msbursts → 300km.

(4). Class of BHs (i.e. formation) and detection (see matter as it falls in)

SMBH	$10^4 - 10^9 M_\odot$ galaxy evolution + mergers	- stellar dynamics - accretion of interstellar gas - disruption + accretion of ☆.
IMBHs	$10^2 - 10^4 M_\odot$ (?)	ULXs ? ??
Stellar BHs	$>3 M_\odot$ collapse of massive ☆	accretion + observe matter just as it falls in - galaxies
primordial BHs	during Big Bang.	?

(5) V. v. brief intro to GR

- In GR, mass curves spacetime (a dynamic entity)

  ⇒ geometry encoded in $\boxed{g_{\mu\nu}}$: the metric.

  cf. in Newt. gravit : flat space + preferred time.

- In Minkowski spacetime (special rel., pseudo-Euc), we can choose coordinates everywhere :

$$\underbrace{ds^2}_{} = dt^2 - \frac{1}{c^2}\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right)$$

ds : 'interval'

→ + physical spacetime between two infinitely separated points separated by $dt, dr, d\theta, d\phi$ @ t, r, θ, ϕ.

+ Lorentz invariant - does not depend on the inertial frame that it is evaluated in

+ $ds^2 > 0$: timelike (light has enough time to travel between two events)

$= 0$: null

< 0 : spacelike.

— In GR, spacetime is curved : (* only choose coordinates locally *)

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

· contains all the info. on curvature of spacetime

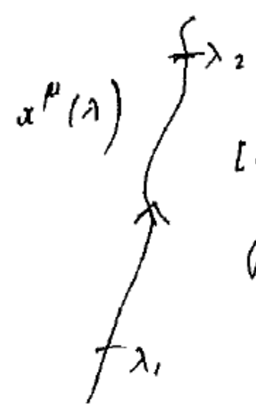
→ ds measures propertime interval $d\tau$ along line : $ds^2 = d\tau^2$.

Newtonian	GR
+ flat space preferred time	curved spacetime
+ Potential ϕ $\nabla^2 \phi = 4\pi G \rho$	+ Ein. Field Eq^{ns} relate geometry & curvature to matter & energy dist^{n} : $\underbrace{G_{\mu\nu}}_{\text{Einstein tensor}}(g, \partial g, \partial\partial g) = \frac{8\pi G}{c^4} \underbrace{T_{\mu\nu}}_{\text{stress matter - energe tensor}}$ 2nd order diff. eq^{n}.
+ $\ddot{\vec{x}} = -\nabla \phi$	particles move on geodesic curve joining 2 events for which interval is <u>extremal</u> [cf. Minkowski space $\rightarrow$ 4D straight lines]

- <u>Geodesic</u> : path a freely falling test particle will take.

→ curve joining 2 events for which interval is <u>extremal</u> (max, min, stationary pt).

i.e. interval is invariant to 1st order in actual path used.



[massive particles $\lambda \sim$ proper time
γs : $\lambda \sim$ affine param].

$$S_{AB} = \int_A^B ds = \int_A^B [g_{\mu\nu}\, dx^\mu dx^\nu]^{1/2} = \int_A^B \left[g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}\right]^{1/2} ds$$

⇒ determined by spacetime geometry

$$\begin{cases} g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu = -1 & \text{massive particles.} \\ \phantom{g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu} = 0 & \gamma\text{s.} \end{cases}$$

Schwarzchild :- describes spherically symmetric gravitational field in vacuum surrounding some massive object.
(1916 : few mnths after GR)

[Birkhoff theorem : spacetime geometry is always Schwarz. outside a general spherical sym. matter].

- c.f. Newtonian : any pt. outside a spherical mass dist^{n} → grav. field depends only on mass interior to pt.

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \frac{1}{c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2$$

$$+ \frac{r^2}{c^2}\left(d\theta^2 + \sin^2\theta \, d\phi^2\right).$$

a) Consider a clock @ rest ($dr = d\theta = d\phi = 0$).

A time interval $\Delta t'$ @ distance r from BH is observed @ $r = \infty$ to be:

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - 2GM/rc^2}}$$

$\boxed{\text{Time Dilation}}$

Clocks run slow for grav. fields.

More importantly, a photon emitted at r $\bar{c}$ freq. ν_e is observed @ ∞ $\bar{c}$ frequency:

$$\nu_\infty = \nu_e \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} \simeq \nu_e \left(1 - \frac{GM}{c^2 r}\right)$$

for $R \gg R_s$ (remember $E \sim h\nu$)

<u>REDSHIFT</u> can be thought of as simply the energy <u>lost</u> to the γ as it climbs out of the potential well.

$$\left[\ 1 + z \approx \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}}\ \right].$$

d) Orbits in Schwarzschild.

- use Euler - Lagrange formalism
- we will consider massive particles (not γs)
  ↳ see S+T, chp 12.

- Remember, in Newtonian $1/r$ potential,

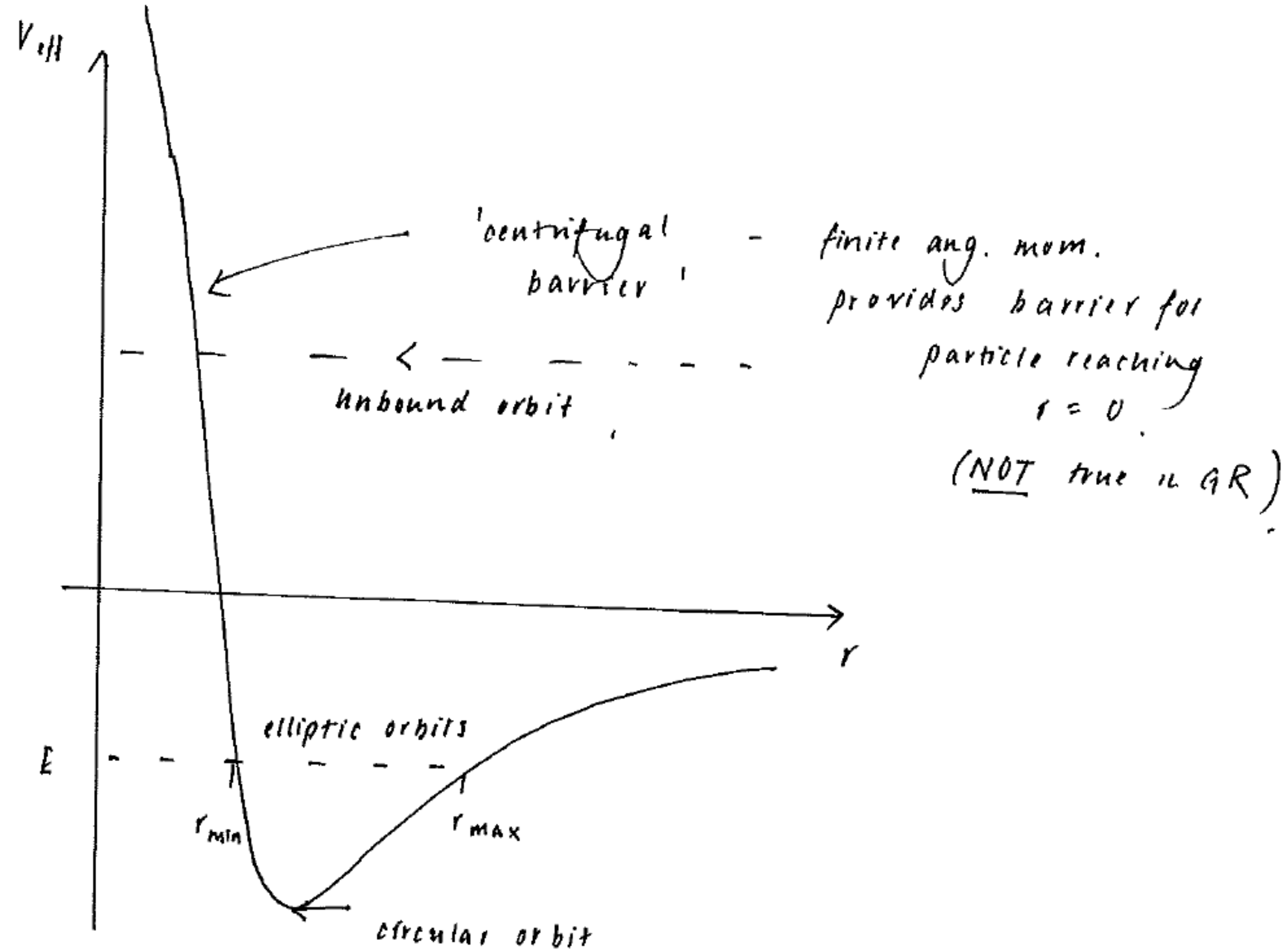
Conservation of Energy: $\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 - \frac{GM}{r} = \underbrace{V_\infty^2}_{E}$

$$\Rightarrow \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + V_{eff} = V_\infty^2 \frac{1}{2}$$

$\bar{c}$ $\quad V_{eff}(r) = -\frac{GM}{r} + \underbrace{\frac{l^2}{2r^2}}_{\text{centrifugal potential}}$ where $l = r^2\dot{\theta}$

~ orbital ang. momentum per unit mass.

for fixed l, , Newtonian grav.



+ Bound orbits for $V_{eff} < 0$

+ circular orbits $\frac{dV_{eff}}{dr} = 0$. $\Rightarrow$ Kepler's 3rd law.

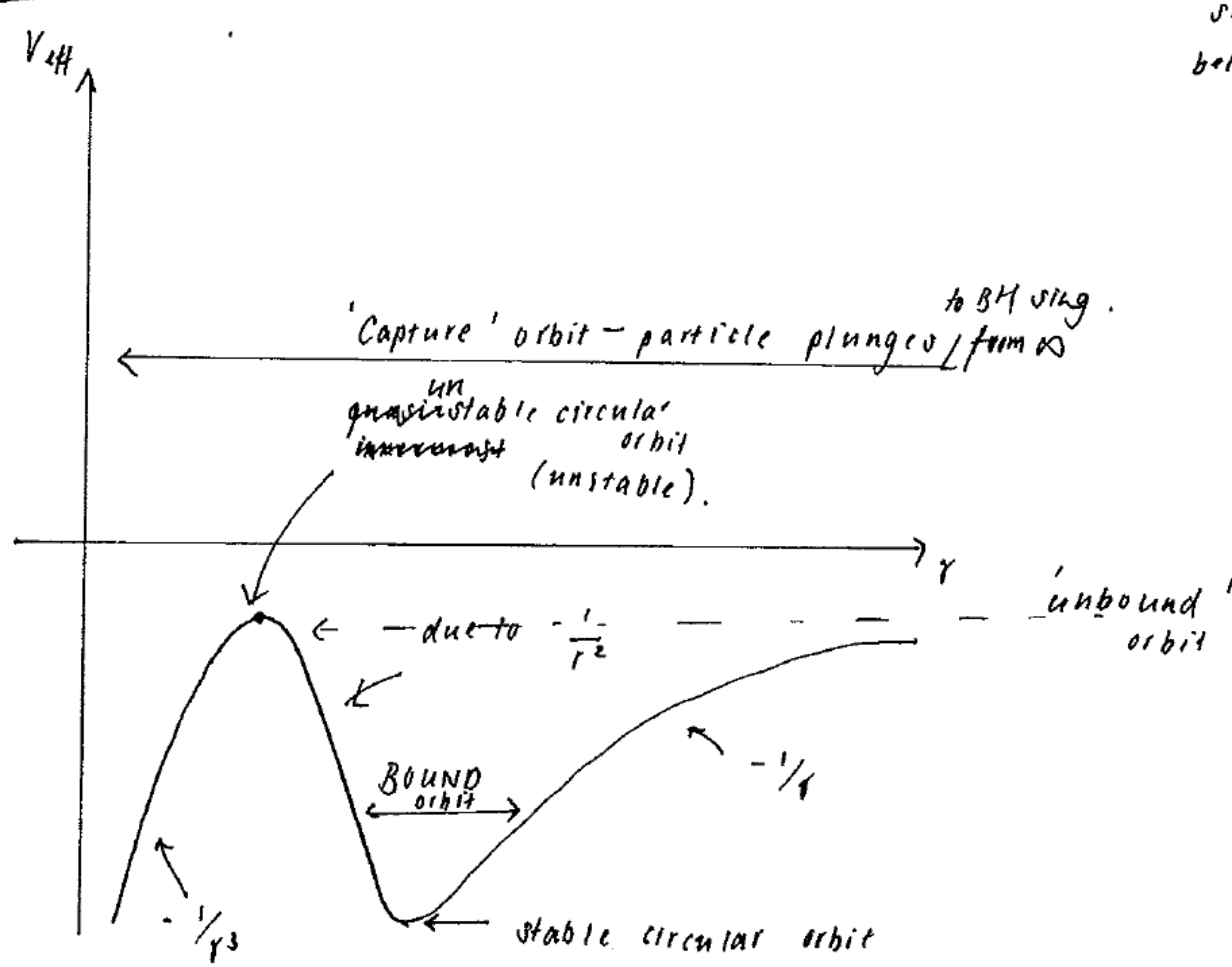
+ Elliptic + hyperbolic orbits.

- In GR, massive particles + Schwarz.

$$V_{eff} = -\frac{GM}{r} + \frac{l^2}{2r^2} - \underbrace{\frac{GMl^2}{c^2 r^3}}$$

GR term - all energy contributes (internal, rot., pot.) to gravitating mass of system.

⇒ for specific l ($l > 2\sqrt{3}$; no maxima/minima when $l < 2\sqrt{3}$ → see below).



Different types of orbits : i) 'capture' - particle plunges into BH.

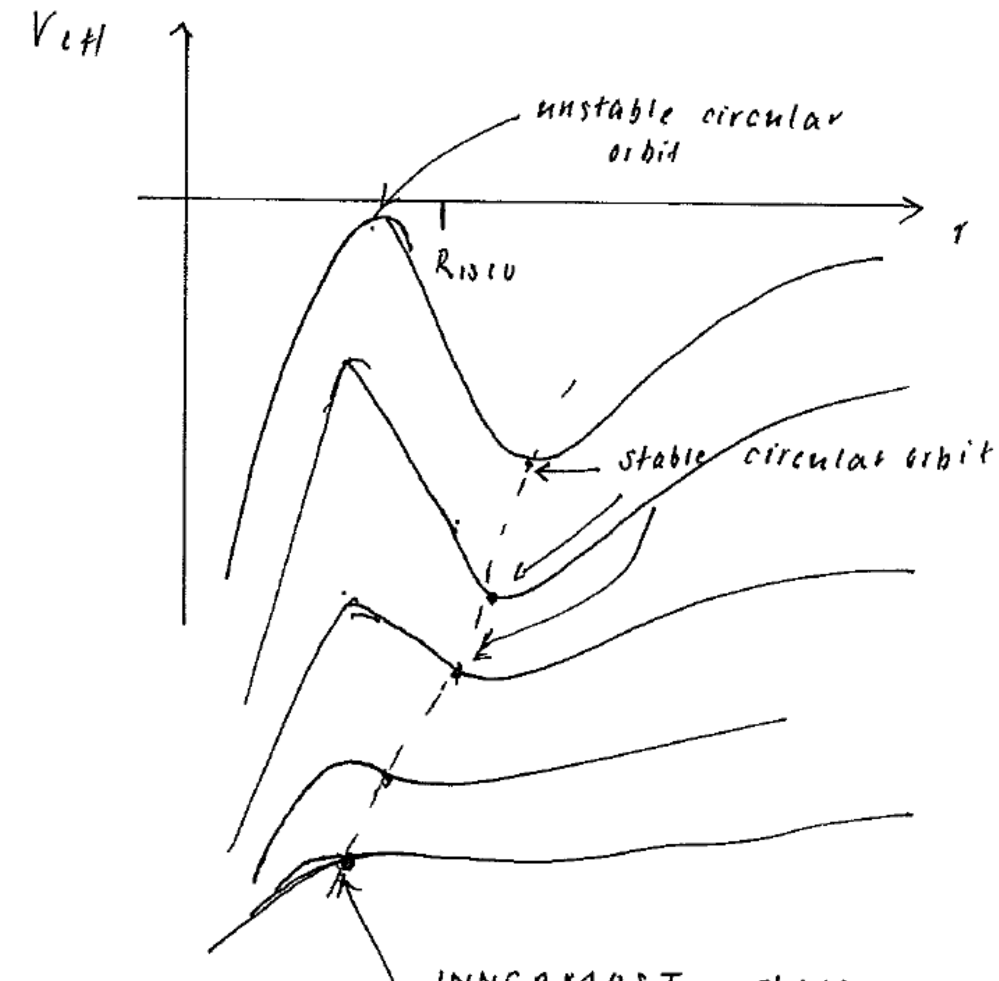
ii) unbound - particle comes from ∞ + returns to ∞

'zoom-whirl'

iii) bound

.... etc ...

[for sequence of ls :



INNERMOST stable circular orbit

@ R_{isco}]

$\frac{dV_{eff}}{dr} = 0$ when $Mr^2 - \ell^2 r + 3M\ell^2 = 0$ $[G = c = 1]$

$$\Rightarrow r = \frac{1}{2M}\left[\ell^2 \pm \sqrt{\ell^4 - 12M\ell^2}\right]$$

$\therefore$ when $\ell \lesssim 2\sqrt{3}\,M$ (no stable circular orbits)

$\Rightarrow \exists$ radius of innermost stable circular orbit (ISCO) is

$$\boxed{r_{ISCO} = 6M = 3R_s} \quad *$$

- smallest radius of any accretion disk around Schwarz. BH.

- Binding Energy for particle per unit mass in last stable orbit : $E_{binding} = \frac{m - E}{m} = 1 - \left(\frac{8}{9}\right)^{1/2}$

$$\approx 5.72\%$$

$\Rightarrow$ frac. of rest mass E released when particle originally @ rest @ ∞

(cf. $\mathcal{E}_{nuclear} \sim 0.7\%$ N.B. $H \xrightarrow{\text{nuclear burn}} He$; 26 MeV per He nucleus)

$\Rightarrow$ Accretion of BHs provides powerful energy source!

[Remember simplistic picture of seq. of V_{eff} for different ls]

$\rightarrow$ gas loses ang. mom. due to disk viscosity (turbulant / magnetic, MRI, ...)

$\rightarrow$ moves inwards $\rightarrow$ grav. P.E increases + heats up

$\rightarrow$ eventually loses enough ang. mom. + no longer follows stable circular orbit + falls into BH.

Part III

Kerr BHs. : 1963 (Kerr-Newman; '65)

a) mass, spin $\underbrace{a}_{\text{ang. mom. per unit mass}} = \frac{J}{M}$ [G = c = 1].

also, define dimensionless $a^* = \frac{J}{M^2}$ $\left(\equiv \frac{Jc}{GM^2}\right.$ using $[G] \sim \frac{cm^3}{g s^2}$ and $[J] \sim \frac{g cm^2}{s}$)

b) $0 < a < \frac{GM}{c^2}$ or $0 < a^* < 1$

∃ Maximally spinning BH (cf. max. ang. mom for a NS beyond which it becomes centrifug. unbound)

⇒ otherwise 'naked singularity' - not clothed by EH.

⇒ any spinning BH will settle down to Kerr BH.

c) Collapse $\star \to$ BH : if ang. mom. is conserved, we expect rapidly spinning BH.

$$L = I\omega = I_\star \omega_\star = I_{BH}\,\omega_{BH}$$

$$\Rightarrow \quad \omega_{BH} \propto \left(\frac{I_\star}{I_{BH}}\right)\omega_\star \propto \left(\frac{R_\star}{R_S}\right)^2 \omega_\star$$

d) <u>Kerr Metric</u> (Boyer-Lindquist coordinates) - 1963, Penrose, Bardeen.

↳ no longer spherically symmetric.

⇒ spinning BH drags spacetime around it.

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}\,dt\,d\phi + \dots$$

TWISTING of spacetime

→ frame dragging in dir$^{\underline{n}}$ of rotation

$$+ \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2$$

where $\Delta = r^2 - 2Mr + a^2$

$$\Sigma = r^2 + a^2\cos^2\theta$$

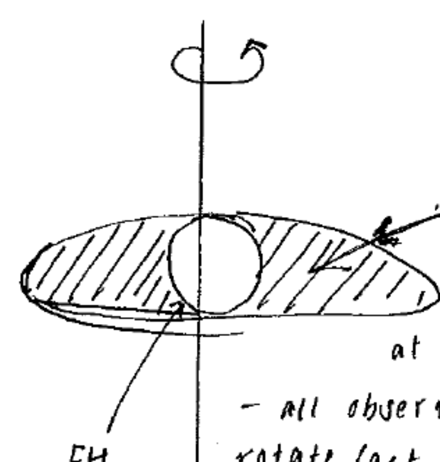
- <u>Event Horizon</u> -

$$\Delta = 0 \Rightarrow R_{EH} = \frac{R_S + \sqrt{R_S^2 - 4a^2}}{2}$$

- <u>Ergosphere</u> - surface where g_{00} flips sign.

$$\Rightarrow R_{EG} = \frac{R_S + \sqrt{R_S^2 - 4a^2\cos^2\theta}}{2}$$

- R_{EH} and R_{EG} $\Rightarrow$ coordinate singularities.



Ergosphere 'Static limit'

→ no object can remain at rest relative to background ☆s

- all observers within ergosphere must rotate (get 'dragged') $\bar{c}$ BH - cannot be

- Stable orbits are possible in EG but only prograde

- $a = 0$; $R_{EH} = R_{EG} = R_S$

- $a \to 1$; $R_{EH} \Rightarrow \frac{R_S}{2}$. Extreme BHs.

  Kerr Grav. Pot. can be deeper !

- Cosmic Censorship (Penrose) $\Rightarrow$ Gravitational Collapse from well-behaved initial conditions never gives rise to naked singularity (i.e. one not clothed by EH)

- Penrose Process $\Rightarrow$ If particle from outside ergosphere enters ergosphere + decays/scatters therein, it is possible that one particle comes out $\bar{c}$ more energy / ang. mom. than it started with i.e. energy maybe extracted from spinning BH upto a fraction,

$$1 - \frac{1}{\sqrt{2}}\left[1 + \left(1 + \left(\frac{J}{J_{max}}\right)^2\right)^{1/2}\right]^{1/2} \lesssim 29\% \text{ of } Mc^2$$

may be extracted

[NB. Bardeen showed this is uninteresting astrophysically for 2 part. decay] cf. Super radiant scat.: part of the wave absorbed & the other part is scattered $\bar{c}$ more energy than incident wave.

- ISCO

		% rest mass energy of accreted particle
$a^* = 0$;	$R_{ISCO} = R_{ISCO,S} = 2M$	5.72 %
$a^* = 1$;	$R_{ISCO} = M$ (prograde)	42.3%
$a^* = -1$;	$R_{ISCO} = 9M$ (retrograde)	3.77 %

- Blandford - Znajek: energy maybe extracted by threading BH with B fields.

Part II: GWs

Outline

- Perturbation Theory: linearised field equations of GR
- Tranverse-Traceless Gauge
- Effects of GWs on freely falling test particles
- Production of GWs
- GW Energy Loss

TABLE 23.1 Production of Linearized Gravitational and Electromagnetic Waves

	Linearized gravitation ($c = G = 1$)	Electromagnetism ($c = 1$)
Field equation	Einstein equation with $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$	Maxwell's equations
Basic potentials	Linearized metric perturbations $h_{\alpha\beta}(x)$	Vector and scalar potentials $(\Phi(x), \vec{A}(x))$
Sources	Stress-energy $T_{\alpha\beta}$	Charge and current $(\rho_{\text{elec}}, \vec{J})$
Lorentz gauge	$\frac{\partial \bar{h}^{\alpha\beta}}{\partial x^\alpha} = 0$	$\frac{\partial \Phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0$
Wave equation with source	$\Box \bar{h}_{ij} = -16\pi T_{ij}$	$\Box \vec{A} = -\mu_0 \vec{J}$
General solution	$\bar{h}^{ij} = 4 \int d^3x' \frac{[T^{ij}]_{\text{ret}}}{\lvert\vec{x} - \vec{x}'\rvert}$	$\vec{A} = \frac{\mu_0}{4\pi} \int d^3x' \frac{[\vec{J}]_{\text{ret}}}{\lvert\vec{x} - \vec{x}'\rvert}$
Large r, long-wavelength approximation	$\bar{h}^{ij} = \frac{2[\ddot{I}^{ij}]_{\text{ret}}}{r}$ $I^{ij} = \int d^3x\, \mu x^i x^j$	$\vec{A} = \frac{\mu_0}{4\pi} \frac{[\dot{\vec{p}}]_{\text{ret}}}{r}$ $\vec{p} = \int d^3x\, \rho_{\text{elec}} \vec{x}$
Time-averaged radiated power	$\frac{dE}{dt} = \frac{1}{5} \langle \dddot{I}_{ij} \dddot{I}^{ij} \rangle$	$\frac{dE}{dt} = \frac{\mu_0}{6\pi} \ddot{\vec{p}}^2$