Part II: GWs

Outline

- Perturbation Theory: linearised field equations of GR
- Tranverse-Traceless Gauge
- Effects of GWs on freely falling test particles
- Production of GWs
- GW Energy Loss

TABLE 23.1 Production of Linearized Gravitational and Electromagnetic Waves

	Linearized gravitation ($c = G = 1$)	Electromagnetism ($c = 1$)
Field equation	Einstein equation with $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$	Maxwell's equations
Basic potentials	Linearized metric perturbations $h_{\alpha\beta}(x)$	Vector and scalar potentials $(\Phi(x), \vec{A}(x))$
Sources	Stress-energy $T_{\alpha\beta}$	Charge and current $(\rho_{\text{elec}}, \vec{J})$
Lorentz gauge	$\frac{\partial \bar{h}^{\alpha\beta}}{\partial x^\alpha} = 0$	$\frac{\partial \Phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0$
Wave equation with source	$\Box \bar{h}_{ij} = -16\pi T_{ij}$	$\Box \vec{A} = -\mu_0 \vec{J}$
General solution	$\bar{h}^{ij} = 4 \int d^3x' \frac{[T^{ij}]_{\text{ret}}}{\lvert \vec{x} - \vec{x}' \rvert}$	$\vec{A} = \frac{\mu_0}{4\pi} \int d^3x' \frac{[\vec{J}]_{\text{ret}}}{\lvert \vec{x} - \vec{x}' \rvert}$
Large r, long-wavelength approximation	$\bar{h}^{ij} = \frac{2[\ddot{I}^{ij}]_{\text{ret}}}{r}$ $I^{ij} = \int d^3x\, \mu x^i x^j$	$\vec{A} = \frac{\mu_0}{4\pi} \frac{[\dot{\vec{p}}]_{\text{ret}}}{r}$ $\vec{p} = \int d^3x\, \rho_{\text{elec}} \vec{x}$
Time-averaged radiated power	$\frac{dE}{dt} = \frac{1}{5} \langle \dddot{I}_{ij} \dddot{I}^{ij} \rangle$	$\frac{dE}{dt} = \frac{\mu_0}{6\pi} \ddot{\vec{p}}^2$

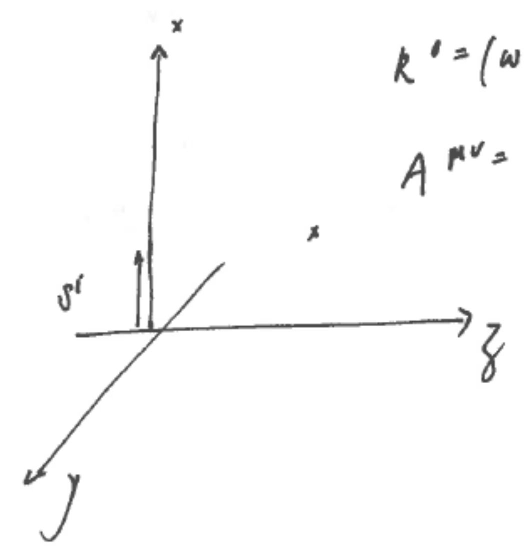
Effect of GWs

Consider a local inertial frame of a test particle with a second test particle separated by S^μ:

Effects of GWs

Separation : $S^\mu = (S^0, \vec{S})$

$S^\mu = S^\mu(0) + \delta S^\mu$



$k^\mu = (\omega, 0, 0, \omega)$

$A^{\mu\nu} = \begin{pmatrix} 0 & & \\ & h_+ & h_\times \\ & h_\times & h_+ \end{pmatrix}$

Derivation

Geodesic eqn : $\frac{D^2}{D\tau^2} S^\mu = R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho S^\sigma$

Effect of GWs II

Particles @ rest : $u^\nu = (1, 0, 0, 0)$ [to 0th order since $R \sim O(h)$]

So $$\frac{d^2}{dt^2} S^\mu = R^\mu{}_{00\sigma} S^\sigma$$

Calculate: $R^\mu{}_{00\sigma} = \frac{1}{2} \partial_0 \partial_0 h^\mu{}_\sigma$ (TT)

$$\Rightarrow \boxed{\ddot{S}^\mu = \frac{1}{2} \ddot{h}^\mu{}_\sigma S^\sigma}$$

Effect of GWs II

Note $S^0 = \text{const}$, $S^3 = \text{const}$ (no longitudinal mode)

h_+:

$$\delta S^1 = \frac{1}{2} h_+ e^{-i\omega t + i\omega z} S^1(0)$$

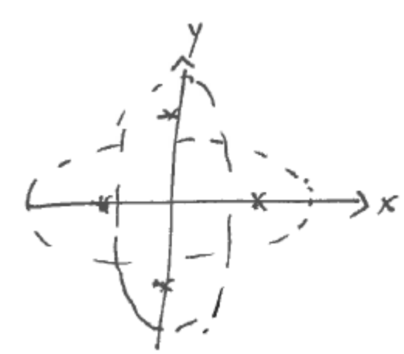
$$\delta S^2 = -\frac{1}{2} h_+ e^{-i\omega t + i\omega z} S^2(0)$$

$h_\times$:

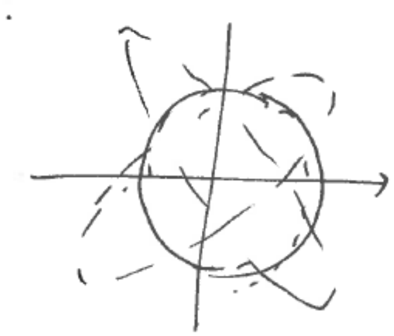
$$\delta S^1 = \frac{1}{2} h_\times e^{-i\omega t + i\omega z} S^2(0)$$

$$\delta S^2 = \frac{1}{2} h_\times e^{-i\omega t + i\omega z} S^1(0)$$

h_+ :



$h_\times$:



Energy of GWs

~ 0th order: object with internal forces, the GW strain acts as an oscillating driving force for mass element m separated by displacement ξ_k

$$F_j = \frac{1}{2} m \ddot{h}_{jk}^{TT} \xi_k$$

GWs act as a force -> must carry energy and angular momentum localised within a few wavelengths (not point!!!):

$$T^{00} = \frac{T^{0z}}{c} = \frac{T^{zz}}{c^2} = \frac{1}{16\pi} \frac{c^2}{G} \langle (\dot{h}_+)^2 + (\dot{h}_\times)^2 \rangle,$$

energy

energy flux

momentum flux

Generation of E&M

Leading order multipole radiation for non-relativistic charge distribution:

$$A_j(t, \mathbf{x}) = \frac{1}{cr}\dot{d}_j\left(t - \frac{r}{c}\right),$$

As 1/r electric and magnetic fields depend only on components of **d** transverse to the propagation direction **n**

$$d_j^T \equiv P_{jk} d_k,$$

where P_{jk} is the projection tensor,

$$P_{jk} \equiv \delta_{jk} - n_j n_k.$$

Luminosity EM

Angular distribution for energy flux:

$$\frac{d^2E}{dt\,d\Omega} = \frac{1}{4\pi c^3}\ddot{d}_j^T \ddot{d}_j^T$$

$$= \frac{1}{4\pi c^3}\left[\ddot{d}_j\ddot{d}_j - \left(n_j\ddot{d}_j\right)^2\right].$$

Integrating over solid angle:

$$L_{\rm em} \equiv \frac{dE}{dt} = \frac{2}{3c^3}\ddot{d}_j\ddot{d}_j.$$

Generation of GWs

Leading order gravitational radiation for quadrupole mass distribution:

$$h_{jk}^{TT}(t, \mathbf{x}) = \frac{2}{r} \frac{G}{c^4} \ddot{\mathcal{I}}_{jk}^{TT}\left(t - \frac{r}{c}\right),$$

where $\mathcal{I}_{jk}$ is the mass quadrupole moment

$$\mathcal{I}_{jk} \equiv \sum_A m_A \left[x_j^A x_k^A - \frac{1}{3} \delta_{jk} (x^A)^2 \right].$$

Taking transverse-traceless part: $\mathcal{I}_{jk}^{TT} \equiv P_{jl} P_{km} \mathcal{I}_{lm} - \frac{1}{2} P_{jk} (P_{lm} \mathcal{I}_{lm}).$

ORDER OF MAGNITUDE: $h \sim \frac{r_{\text{Sch}}}{r} \frac{v^2}{c^2},$

Returning to luminosity of GWs

Energy flux is given by stress-matter energy tensor:

$$T_{0r} = \frac{1}{32\pi}\frac{c^4}{G}\langle h^{TT}_{jk,0} h^{TT}_{jk,r}\rangle$$

$$\frac{d^2E}{dt\,d\Omega} = \frac{1}{8\pi}\frac{G}{c^5}\langle \dddot{I\!\!\!-}^{TT}_{jk}\dddot{I\!\!\!-}^{TT}_{jk}\rangle$$

$$= \frac{1}{8\pi}\frac{G}{c^5}\left\langle \dddot{I\!\!\!-}_{jk}\dddot{I\!\!\!-}_{jk} - 2n_i\dddot{I\!\!\!-}_{ij}\dddot{I\!\!\!-}_{jk}n_k + \frac{1}{2}\left(n_j n_k \dddot{I\!\!\!-}_{jk}\right)^2\right\rangle,$$

NEWTONIAN QUADRUPOLE FORMULA

slow motion sources,

weak internal gravity

$$L_{GW} \equiv \frac{dE}{dt} = \frac{1}{5}\frac{G}{c^5}\langle \dddot{I\!\!\!-}_{jk}\dddot{I\!\!\!-}_{jk}\rangle.$$

Energy Balance: Radiation Reaction Force

$$\mathbf{F}^{(\mathrm{react})} = -m\nabla\Phi^{(\mathrm{react})}, \qquad \Phi^{(\mathrm{react})} = \frac{1}{5}\frac{G}{c^5}\,I\!\!\!-_{jk}^{(5)}x_j x_k .$$

$$\begin{aligned}
\frac{dE}{dt} &= \sum_A \mathbf{v}_A \cdot \mathbf{F}_A^{(\mathrm{react})} \\
&= -\sum_A m_A v_{Aj}\frac{2}{5}\frac{G}{c^5}\,I\!\!\!-_{jk}^{(5)}x_k^A \\
&= -\frac{1}{5}\frac{G}{c^5}\,I\!\!\!-_{jk}^{(5)}\frac{d}{dt}\sum_A m_A x_j^A x_k^A \\
&= -\frac{1}{5}\frac{G}{c^5}\,I\!\!\!-_{jk}^{(5)}I\!\!\!-_{jk}^{(1)},
\end{aligned}$$

See Thorne (1980)

Order of magnitude estimate

$$\dddot{\bar{I}}_{jk} \sim \frac{MR^2}{T^3} \sim \frac{Mv^3}{R},$$

COMPACT sources
v ~ c; R ~ r_sch

$$\frac{dE}{dt} \sim \frac{G}{c^5}\left(\frac{M}{R}\right)^2 v^6 \sim L_0\left(\frac{r_{\rm Sch}}{R}\right)^2\left(\frac{v}{c}\right)^6,$$

for a binary system

$$L_0 \equiv \frac{c^5}{G} = 3.6 \times 10^{59}\ \text{erg s}^{-1}.$$

$$L = \frac{+2}{5} \frac{G^4 M^5}{R^5}$$

Radiated energy and peak luminosity

- Using fits from NR simulations for the total energy radiated from infinity and the posterior distributions for masses and spins from the analysis we also infer:
- Radiated energy: $3.0 \pm 0.5 M_{\odot}$
- Peak luminosity: $3.6 \pm 0.4 \times 10^{56}\,\mathrm{erg\,s^{-1}}$

~ Solar luminosity × 10^{23}

~ the visible Universe's galactic luminosity × 50

for a binary system

$$L = \frac{+2}{5} \frac{G^4 M^5}{R^5}$$

Efficiency of GW emission

$$\Delta E = \varepsilon M c^2, \qquad \varepsilon \sim \left(\frac{r_{\mathrm{Sch}}}{R} \right)^{7/2}.$$

see homework questions