Formation and Evolution of Galaxies 2017/2018, Week 1

1.1 Mass distribution of the Milky Way

The rotation curve of the Milky Way (and most disc galaxies) is approximately flat: the velocities of stars on circular orbits in the disc is about constant, $v_c(R) \approx$ const, even at large radii.

a. Assuming that the mass distribution in the Milky Way is dominated by a spherically symmetric dark matter halo, show that a flat rotation curve implies the following density profile:

$$\rho_h(R) = \frac{v_c^2}{4\pi G} R^{-2} \tag{1.1.1}$$

where R is the galactocentric distance and v_c the circular velocity.

b. For a density distribution of the form (1.1.1), how does the gravitational potential depend on R?

c. For a circular velocity $v_c = 200$ km/s, $R_0 = 8$ kpc, calculate the density of the dark matter halo near the Sun. Give the answer in kg m^{-3} and in Solar masses per cubic parsec (one solar mass is $M_\odot = 2 \times 10^{30}$ kg and one parsec $= 3.08 \times 10^{16}$ m, and the gravitational constant is $G = 6.67 \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$).

A density distribution of the form (1.1.1) is known as a *singular isothermal sphere*. While the simple analytical form is appealing, there are two obvious problems: At $R = 0$ the density is infinite, and the total mass is also infinite. Hence, more realistic models are clearly required for a realistic description of the structure of real halos.

1.2 The flatness problem

In this assignment we explore the evolution of the density parameter for matter, Ω_M, with redshift/scale factor. According to the CMB measurements by the Planck satellite, the *current* value is $\Omega_{M,0} = 0.31$ while the dark energy density parameter is $\Omega_{\Lambda,0} = 0.69$, making the Universe exactly flat with $\Omega = 1.0$.

Recall that the critical density, at any epoch, is

$$\rho_c = 3H^2/8\pi G \tag{1.2.1}$$

with $H \equiv \dot{a}/a$ and the time derivative of the scale factor is given by the Friedman equation, which can be written as

$$\dot{a} = H_0 \left[\Omega_{M,0}(1/a - 1) + \Omega_{\Lambda,0}(a^2 - 1) + 1\right]^{1/2} \tag{1.2.2}$$

where we have ignored the energy density due to radiation.

- Now show that regardless of the present-day values of $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$, the Universe approached $\Omega_M = 1$ at high redshift.

- If the matter density is currently $\Omega_{M,0} = 0.3$, then how much did it deviate from unity at $z = 10$? At $z = 1000$?

In the context of cosmology, this is known as the *flatness* problem: Why is the current value of Ω_M close to, but not exactly unity? It requires an exceedingly high degree of fine-tuning to produce the tiny departure from $\Omega_M = 1$ at high redshifts that result in a present-day Universe whose density parameter is neither very different from, nor exactly equal to unity.

From the perspective of Galaxy Formation and Evolution, however, it is very convenient that the Universe behaved in this way until relatively recently. In terms of structure formation, the regime of linear grown occurred under conditions where the density was very close to the critical value and the Ω_Λ term negligible.

1.3 Cosmological distances

Recall that a line element in the Friedman-Robertson-Walker metric may be written as

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)\right] \tag{1.3.1}$$

for scale factor a, radial coordinate r, and curvature K.

It is useful to define the *angular diameter distance*, d_A, as

$$d_A = \frac{a_0 r}{1 + z} \tag{1.3.2}$$

where a_0 is the current value of the scale factor. With this definition, we then have following relation between the length dl of a standard rod, oriented perpendicular to the line-of-sight, the apparent angular size of the rod $d\theta$, and d_A:

$$dl = d_A d\theta \tag{1.3.3}$$

which is similar to the usual Euclidian relation.

For a photon emitted from a source at $t = t_1$ and received by an observer at $t = t_0$, the co-moving distance to the source is

$$\chi = \int_{t_1}^{t_0} \frac{c}{a(t)} dt \tag{1.3.4}$$

In general, this expression must be integrated numerically, although analytic solutions are possible in some cases. Here we explore one such case, the *Einstein-de Sitter Universe*.

In an Einstein-de Sitter Universe, $\Omega_0 = 1$ and $\Omega_\Lambda = 0$, and the co-moving radial coordinate is proportional to the comoving distance: $\chi = a_0 r$. It is customary to set $a_0 = 1$. For this particular case, the cosmic time t, the Hubble constant H_0, and the scale factor $a(t)$ are related as:

$$a(t) = \left(\frac{3H_0 t}{2}\right)^{2/3} \tag{1.3.5}$$

a. Show that, for an Einstein-de Sitter Universe, the comoving radial coordinate r and the redshift z are related as

$$r = \frac{2c}{H_0}\left(1 - (1 + z)^{-1/2}\right) \tag{1.3.6}$$

Hint: The following integral may come in handy:

$$\int_0^a (1 + x)^{-3/2} \, dx = 2\left(1 - (1 + a)^{-1/2}\right) \tag{1.3.7}$$

b. Show that d_A has an extremum at $z = 5/4$ in the Einstein-de Sitter Universe, and argue that this must be a maximum. What does this imply for the apparent sizes of objects (of a given linear size) as a function of redshift?