# Formation and Evolution of Galaxies 2017/2018, Week 2

## 2.1 Cosmological surface brightness dimming

In astronomy, the *luminosity* $L$ of a source is the energy output per unit time (e.g. measured in W), the *flux* is the energy passing through a surface of unit area per unit time (e.g. in units of W m$^{-2}$) and the *intensity* $I$ of radiation is the flux per unit solid angle (W m$^{-2}$ sr$^{-1}$). It is straight forward to show that the intensity is distance-independent in standard Euclidian geometry, as long as there is no absorbing material between the source and the observer.

- Using the definitions of angular diameter- and luminosity distance, show that the intensity of a cosmological source decreases with redshift as

  $$I(z) = I_0(1+z)^{-4} \tag{2.1.1}$$

  where $I_0$ is the intensity that would be measured at $z = 0$.

## 2.2 Classical interpretation of Friedmann equation

Consider a thin mass shell surrounding a spherically symmetric of mass, $M$, with radius $r$. From simple energy considerations, the velocity and radius of the shell are related to the total energy $\mathcal{E}$ as

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{GM}{r} = \mathcal{E} \tag{2.2.1}$$

Compare this with the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3} \tag{2.2.2}$$

and show that (2.2.1) resembles the Friedmann equation without the cosmological constant term. How is the total energy related to the curvature?

## 2.3 Virial theorem for uniform sphere

For any self-gravitating system, the *virial theorem* states that

$$E_{\mathrm{kin}} = -\frac{1}{2}E_{\mathrm{pot}} \tag{2.3.1}$$

In this exercise we will see how the virial theorem provides a way to determine the mass of a structure in virial equilbrium from measurements of the velocity dispersion and radius.

The total kinetic energy is

$$E_{\mathrm{kin}} = \frac{1}{2}M\sigma^2 \tag{2.3.2}$$

where $M$ is the mass and the velocity dispersion is

$$\sigma \equiv \frac{1}{N-1}\sum_i |\mathbf{v} - \langle\mathbf{v}\rangle|^2 \tag{2.3.3}$$

For an isotropic velocity distribution, $\sigma^2 = 3\sigma_r^2$ where $\sigma_r^2$ is the velocity dispersion measured along a particular line of sight. We then have

$$M = -\frac{E_{\mathrm{pot}}}{3\sigma_r^2} \tag{2.3.4}$$

We now need to establish how $E_{\mathrm{pot}}$ is related to the mass and radius of the structure.

**a.** Show that, for a uniform (constant density) sphere of mass $M$ and radius $R$, the potential energy is

$$E_{\text{pot}} = -\frac{3GM^2}{5R} \tag{2.3.5}$$

*Hint:* start by considering the potential energy of a thin shell with density $\rho$ and thickness $\mathrm{d}r$ at radius $r$ within the sphere. Then integrate over all these shells to get the total $E_{\text{pot}}$.

**b.** Thus, show that for a uniform sphere,

$$M = 5\frac{R\sigma_r^2}{G} \tag{2.3.6}$$

Eq. (2.3.6) can be applied to gravitationally bound systems in virial equilibrium, for which we can measure $\sigma$ and $R$. An example of such a system is a *galaxy cluster*. Indeed, the first strong evidence for *dark matter* is due to Fritz Zwicky, who (in 1933) compared measurements of the mass of the Coma galaxy cluster with the amount of visible matter in the cluster, and from this concluded that large amounts of unseen matter must be present in the cluster.

A galaxy cluster, like most astrophysical objects, is clearly not a uniform sphere. Furthermore, for more realistic mass distributions, it is often difficult to measure the "total" radius. However, for a fairly wide range of assumptions about the actual profile, the mass can be estimated as

$$M = 10\frac{R_h\sigma_r^2}{G} \tag{2.3.7}$$

where $R_h$ is the half-mass radius.

**c.** For the Coma cluster, measurements of the radial velocities of galaxies in the cluster yield $\sigma_r = 1000$ km s$^{-1}$. The average radial velocity is $\langle v \rangle = 6900$ km s$^{-1}$ and the half-light radius is about 0.7°. Assuming a value of $H_0 = 72$ km s Mpc$^{-1}$ for the Hubble constant, what is your best estimate of the mass of the cluster? Which additional assumption(s) do you need to make?

Constants:
Solar mass: $M_\odot = 2 \times 10^{30}$ kg
Parsec: 1 pc = $3.08 \times 10^{16}$ m
Gravitational constant: $G = 6.67 \times 10^{-11}$ m$^3$ kg$^{-1}$ s$^{-2}$

## 2.4 Virial radius

In the first set of assignments we encountered the singular isothermal sphere, with a density profile

$$\rho = \frac{v_c^2}{4\pi G r^2} \tag{2.4.1}$$

We have seen in the lecture that virialised structures have overdensities $\Delta \equiv \rho_h / \langle \rho \rangle \gtrsim 200$ where $\rho_h$ is the density of the halo and $\langle \rho \rangle$ is the mean cosmic density of matter, $\Omega_m \rho_{\rm crit}$. Since the density of a profile of the form (2.4.1) decreases outwards, we can always define some *virial radius* $r_{\rm vir}$ within which the mean halo density exceeds the mean density of the Universe by a given factor (e.g. $\Delta = 200$).

- Show that the virial radius is given by

  $$r_{\rm vir} = \sqrt{\frac{200}{\Delta_{\rm vir}\Omega_m}} \frac{v_c}{10H} \tag{2.4.2}$$

  (Equation 7.137 in MBW).