Werkcollege, Cosmology 2014/2015, Week 4

These are the exercises and hand-in assignment for the 4th week of the course *Cosmology*. Every week, one of the problems provides credit towards the final exam. If at least **9** of these problems are handed in and approved, one problem on the final exam may be skipped. The hand-in assignment for this week is **Problem 4.2** below.

Problem 4.1 - Cosmological surface brightness dimming

In astronomy, the *luminosity* L of a source is the energy output per unit time (e.g. measured in W), the *flux* is the energy passing through a surface of unit area per unit time (e.g. in units of W m^{-2}) and the *intensity* I of radiation is the flux per unit solid angle (W m^{-2} sr^{-1}). It is straight forward to show that the intensity is distance-independent in standard Euclidian geometry, as long as there is no absorbing material between the source and the observer.

- Using the definitions of angular diameter- and luminosity distance, show that the intensity of a source decreases with redshift as

$$I(z) = I_0(1+z)^{-4} \tag{4.1.1}$$

Problem 4.2 - Cosmological distances (adapted from Reexam 2013/2014)

Recall that a line element in the Friedman-Robertson-Walker metric may be written as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{a^2(t)}{c^2}\left[\mathrm{d}r^2 + \mathcal{R}^2\sin^2(r/\mathcal{R})(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] \tag{4.2.1}$$

for scale factor a, co-moving radial coordinate r, and radius of curvature $\mathcal{R}$.

We have seen that it is useful to define the *angular diameter distance*, D_A, as

$$D_A = \frac{D}{1+z} \tag{4.2.2}$$

for distance measure

$$D = \mathcal{R}\sin(r/\mathcal{R}). \tag{4.2.3}$$

With this definition, we then have following relation between the length $\mathrm{d}l$ of a standard rod, oriented perpendicular to the line-of-sight, the apparent angular size of the rod $\mathrm{d}\theta$, and D_A:

$$\mathrm{d}l = D_A\mathrm{d}\theta \tag{4.2.4}$$

which is similar to the usual Euclidian relation.

The general expression for the comoving radial coordinate, r, is

$$r = \int_{t_1}^{t_0} \frac{c}{a(t)}\mathrm{d}t \tag{4.2.5}$$

for light emitted from a source at $t = t_1$ and received by an observer at $t = t_0$. In general, this expression must be integrated numerically, although analytic solutions are possible in some cases. Here we explore one such case, the *Einstein-de Sitter Universe*.

In an *Einstein-de Sitter Universe*, $\Omega_0 = 1$ and $\Omega_\Lambda = 0$. For this particular case, the cosmic time t, the Hubble constant H_0, and the scale factor $a(t)$ are related as:

$$a(t) = \left(\frac{3H_0 t}{2}\right)^{2/3} \tag{4.2.6}$$

4.2.a Show that, for an Einstein-de Sitter Universe, the comoving radial coordinate r and the redshift z are related as

$$r = \frac{2c}{H_0}\left(1 - (1+z)^{-1/2}\right) \tag{4.2.7}$$

Hint: The following integral may come in handy:

$$\int_0^a (1+x)^{-3/2}\,\mathrm{d}x = 2\left(1 - (1+a)^{-1/2}\right) \tag{4.2.8}$$

4.2.b Show that D_A has an extremum at $z = 5/4$ in the Einstein-de Sitter Universe, and argue that this must be a maximum. What does this imply for the apparent sizes of objects (of a given linear size) as a function of redshift?

Problem 4.3 - K-corrections (adapted from Exam 2012/2013)

The relations between observed flux (F) and source luminosity (L) derived in the lecture assume that the observer can detect photons over the full wavelength/frequency interval emitted by the source. In practice, this is almost never the case. Observers instead measure the flux density integrated over some specific wavelength/frequency range. Because of the redshift, the observed wavelength interval will, in general, be different than the wavelength interval over which the photons were emitted. In order to determine luminosity distances, a correction for this effect must therefore be made.

The K-correction is the difference between the observed magnitude $m_{\mathrm{obs}}(z)$ for a source at redshift z and the magnitude that would be observed if the source were at rest, m_{rest}:

$$m_{\mathrm{rest}} = -2.5\log_{10}\int f(\lambda)S(\lambda)\,\mathrm{d}\lambda + \mathrm{const} \tag{4.3.1}$$

$$m_{\mathrm{obs}}(z) = -2.5\log_{10}\int f(\lambda')S[\lambda'(1+z)]\,\mathrm{d}\lambda' + \mathrm{const} \tag{4.3.2}$$

In addition to the redshift z, the K-correction depends on the spectrum of the source (here expressed as a function of wavelength, $f[\lambda]$) and the spectral response of the system used for the observations, $S(\lambda)$. The K-correction is a purely instrumental effect that simply accounts for the fact that light emitted at wavelength λ' is observed at wavelength λ. It does *not* take into account the cosmological effects of the redshift due to the expansion of the Universe.

4.3.a Show that (4.3.1) and (4.3.2) lead to the following expression for the K-correction:

$$K = m_{\mathrm{obs}} - m_{\mathrm{rest}} \tag{4.3.3}$$

$$= 2.5\log_{10}\frac{\int f(\lambda)S(\lambda)\,\mathrm{d}\lambda}{\int f[\lambda/(1+z)]S(\lambda)\,\mathrm{d}\lambda} + 2.5\log_{10}(1+z) \tag{4.3.4}$$

4.3.b Find and write down the equivalent expression for the K-correction in terms of the spectrum as a function of *frequency*, $f(\nu)$

4.3.c Calculate the K-correction for a source with a power-law spectrum, $f(\lambda) \propto \lambda^{\beta}$. To simplify the calculations, you can approximate the bandpass transmission curve $S(\lambda)$ as a box function, i.e., assume that $S(\lambda)$ is a (positive) constant for $\lambda_1 < \lambda < \lambda_2$ and zero elsewhere.